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The evolution of consistent conjectures

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Abstract

In this paper we model the evolution of conjectures in an economy consisting of a large number of firms which meet in duopolies. The duopoly game is modelled by the conjectural variation (CV) model. An evolutionary process leads to more profitable conjectures becoming more common (payoff-monotonic dynamics). Under this class of selection dynamics, convergence occurs to a small set of serially undominated strategies containing the consistent conjecture. This set can be made arbitrarily small by appropriate choice of the strategy set. If the game is dominance solvable, then the dynamics converge globally to the unique attractor.
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1. Introduction

Beliefs determine behavior; behavior determines payoff. From an evolutionary perspective, those types of behavior that lead to higher payoffs tend to become more common. Why not apply the same argument to the beliefs that give rise to those forms of behavior? We call this the *evolutionary* approach to explaining beliefs. This contrasts with the *epistemic* approach that seeks to explain beliefs in terms of rationality and the information available to agents. The evolutionary approach explains beliefs in the context of bounded rationality.

We explore the evolutionary approach to beliefs in the conjectural variations (*CV*) model, which has a long history in industrial organization and oligopoly theory but has been subject to much recent criticism by game theorists. We discuss this in detail in Section 2. Not only has

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the CV approach proven useful in a wide range of theoretical and empirical applications, but it has also been reappraised in several more recent theoretical papers (Driskill and McCafferty, 1989; Dockner, 1992; Cabral, 1995).

In the CV approach, the duopolistic firm has a belief: its conjecture about how the other firm will respond if it alters its own output (the conjecture ϕ is the derivative of the other firm's output with respect to own-output). This belief gives rise to behavior: the firm's beliefs about its competitor will determine its own maximizing choice of output in relation to its competitor's. This behavior is captured in the decision rule (reaction function). We consider the particularly simple case of homogeneous good Cournot duopoly with quadratic costs of production. In this case, the CV model yields linear decision rules with a unique mapping from beliefs to the decision rules: each decision rule can be thought of as representing or embodying the belief of the firm. In its most general form, the conjecture ϕ can be thought of as a vision or belief about competition underlying the corporate strategy (decision rule) of the firm.

We consider an economy populated by firms playing the duopoly game in randomly matched pairs. A process of social evolution occurs, meaning that beliefs that yield more profitable behavior in the competitive process will become more common. The nature of the social evolution can be viewed in several ways. It can be a process of imitation as less successful firms imitate the more successful. It can be a process of propagation, in that the best practice of successful firms is spread by some mechanism: the successful firms diversify (multiply) or the managers of successful firms move and bring the ideas of the successful firm to the less successful firm types. It can also be a method of selection: the least successful firms are more likely to go bankrupt than the more successful. We do not attempt to model this process in detail. Rather, we assume that there is indeed some selection dynamics at work that leads to beliefs that yield a higher payoff becoming more common (*payoff-monotonic dynamics*).

The results we obtain are very clear cut. If we restrict conjectures to a finite set, we find that the only beliefs that survive in the long-run are *close* to the *consistent* conjecture (Proposition 5). *Consistency* here means that the conjecture of a firm about the slope of its competitor's reaction function is equal to the *actual* slope.[2] The surviving conjectures can be made arbitrarily close to the consistent conjecture by making the distance between conjectures small enough. The set of surviving strategies corresponds to the set of serially undominated strategies.[3] If there is a unique conjecture in this set, then it is a global attractor to the payoff-monotonic selection dynamics. In the case where we allow for a continuous strategy set, the consistent conjecture is generally the unique evolutionary stable strategy (Proposition 1).

This is an interesting result: the original literature on consistent or "rational" conjectures saw the justification in terms of epistemic rationality: "rational" firms ought to be correct. Here, however, we can see an alternative explanation: a population of boundedly rational

[2] This is equivalent to Bresnahan's definition of consistency (Bresnahan, 1981). Other early contributions to the literature on consistent conjectures include (Hahn, 1977, 1978; Perry, 1982; Kamien and Schwartz, 1983; Ulph, 1983; Boyer and Moreaux, 1983).

[3] Samuelson and Zhang (1992) A serially undominated strategy is a pure-strategy that survives the iterative elimination of strategies that are strictly dominated by another pure-strategy.

firms can evolve to the consistent conjectures. The criticism of the concept of consistency of conjectures has been even stronger than of CVs in general (see, for example, Makowsky, 1987; Shapiro, 1989; Lindh, 1992). However, this criticism does not apply to the approach adopted in this work, since the criticisms view consistency from a conventional theoretical perspective. From an evolutionary perspective, however, the emergence of consistent or near consistent conjectures is simply a result of selection over time.

2. Conjectural variation duopoly

> There is every reason to think that oligopolists in different markets interact in different ways, and it is useful to have models that can capture a wide range of such interactions. Conjectural oligopoly models, in any event, have been more useful than game-theoretic oligopoly models in guiding the specification of empirical research in industrial economics. (Martin, 1993, p. 30).

We consider an economy populated at any moment by a large number of firms matched in pairs playing a duopoly. We model the duopoly using the notion of *conjectural variations*.

Firms choose output levels which are produced with total cost $\mathrm{TC}^i = (c/2)q^{i2}, i = a, b$. The market price is a linear function of the two outputs $p = 1 - q^i - q^j$. Firm i's payoff function is given by

$$\pi^i \left(q^i, q^j\right) = q^i \left(1 - q^i - q^j - \tfrac{1}{2}cq^i\right). \tag{1}$$

Firms choose output levels given a conjecture about the response of their rival to a change in their level of production. We define such a conjecture as the conjectural variation parameter ϕ^i:

$$\phi^i \equiv \frac{\mathrm{d}q^j}{\mathrm{d}q^i}. \tag{2}$$

The maximization of the profit function (1), given ϕ^i (2) yields the first order condition:

$$\frac{\mathrm{d}\pi^i}{\mathrm{d}q^i} = 1 - [(2 + \phi^i + c)q^i + q^j] = 0. \tag{3}$$

Eq. (3) defines the *decision rule* (DR) of firm i given its belief ϕ^i, which relates the output of firm i to the output of its competitor. The decision rule is linear and parametrized by the intercept term h_0 and slope term h_1, where

$$q^i = h_0^i + h_1^i q^j$$

$$h_0^i = \frac{1}{2 + \phi^i + c}; \qquad h_1^i = -\frac{1}{2 + \phi^i + c}.$$

Firm *behavior* is described by the decision rule $\{h_0^i, h_1^i\}$ and hence the *belief* ϕ. The conjectural variation ϕ is usually interpreted[4] as a measure of the expected competitiveness

[4] Martin (1993, p. 25).

of the rival, ranging from a more collusive conjecture ($\phi > 0$) to a more competitive conjecture ($\phi < 0$). In the rest of the paper we will use the terms conjecture and decision rule (DR) interchangeably.

Given a pair of DRs, the equilibrium output pair is given by the point of intersection. We restrict conjectures to the range[5] $\phi \in [-1, 1]$, capturing all of the economically interesting cases. The range of DRs runs from the Walrasian whereby the firm produces output up to the point where price equals marginal costs ($\phi = -1$) to the perfectly collusive ($\phi = 1$).

It is easy to verify that for pairs of conjectures between $[-1, 1]$, a stage-game equilibrium always exists and is stable when $c > 0$. When $c = 0$, the stage-game is stable unless both firms have conjectures equal to minus one. This allow us to compute the equilibrium pair $\{q^i, q^j\}$ and the related payoffs as a function of firm's beliefs ϕ:

$$q^i\left(\phi^i, \phi^j\right) = \frac{1 + \phi^j + c}{3 + 4c + c^2 + (2 + c)(\phi^i + \phi^j) + \phi^i\phi^j}; \tag{4}$$

$$\pi^i(\phi^i, \phi^j) = \frac{(1 + \phi^j + c)^2(2 + c + 2\phi^i)}{2(3 + 4c + c^2 + (2 + c)(\phi^i + \phi^j) + \phi^j\phi^i)^2}. \tag{5}$$

Expressions (4) and (5) give the equilibrium output and profits given any two DRs $\{\phi^i, \phi^j\}$, respectively. If both firms have $\phi = 1$, the stage-game equilibrium will be symmetric and joint-profit maximizing. When both firms have $\phi = -1$, equilibrium will occur at the Walrasian outcome. If both firms have Cournot conjectures $\phi = 0$, the equilibrium will occur at the Cournot outcome.

The consistent conjecture ϕ^* is defined as in Bresnahan. If both firms hold the same conjecture, it is *consistent* if the actual slope of the DR equals the conjectured slope. That is, ϕ^* is the solution of the h_1^i equation when both h_1^i and ϕ are replaced by ϕ^*. The solution is

$$\phi^* = -\left(1 + \frac{c}{2}\right) + \sqrt{\frac{4c + c^2}{4}} \tag{6}$$

The consistent conjecture is always strictly negative (for $c < \infty$), becoming more negative as c falls, until $\phi^* = -1$ when $c = 0$.

3. The evolution of conjectures

Individual firms in any one period can only adopt a pure-strategy. The finite and ordered strategy set common to all firms is defined as

$$\Phi \equiv \{\phi_i : \phi_i \in [-1, 1], \qquad i = 1, \ldots, n\}.$$

For example, Φ can be a grid of granularity δ so that $n = 1 + (2/\delta)$:

$$\Phi \equiv \left\{\phi_i : \phi_i = -1 + \delta i, \qquad i = 0, \ldots, \frac{2}{\delta}\right\}$$

[5] In fact all, of the results of this paper apply if we restrict attention to conjectures in the range $[-1, k]$ where $k \geq 0$.

When two firms meet their payoff will depend on the specific DRs they employ: let $\pi_{i,j}$ be the payoff when DR_i plays DR_j so that the $n \times n$ payoff matrix is $\boldsymbol{T} \equiv [\pi_{i,j}]$, where $i, j = 1, 2, \ldots, n$, denote row and column, respectively. The distribution of the population across DRs is summarized by a state n-vector $\boldsymbol{z}(t) \equiv \{z_1(t), z_2(t), \ldots, z_n(t)\}$, where $z_i(t)$ represents the share of the population adopting DR i at time t.[6] The set C at time t is the set of conjectures with strictly positive population shares (e.g. it excludes those conjectures which are not being adopted by any firm). This is called the *support* of $\boldsymbol{z}(t)$ and is defined as $C(\boldsymbol{z}(t)) = \{\phi_i \in \Phi : z_i > 0\}$.

Given a population distribution $\boldsymbol{z}(t)$, we can define the average[7] payoff of conjecture ϕ_i as

$$\pi(\phi_i, \boldsymbol{z}) = \sum_{j=1}^{n} z_j \pi_{ij}.$$

The average payoff of a particular conjecture depends on the distribution of conjectures.

The selection dynamics gives the evolution of the distribution of conjectures over time. We adopt the standard forms:

$$\dot{z}_i = g_i(\boldsymbol{z}) z_i \tag{7}$$

for continuous time, or,

$$z_{i,t+1} = z_{i,t} + g_i(\boldsymbol{z}(t)) z_{i,t} \tag{8}$$

for discrete time. In both cases, however, the function $g_i(\cdot)$ specifies the rate at which pure-strategy i (DR_i) replicates when the population is in state $\boldsymbol{z}$. We focus on a particular class of dynamics called *payoff-monotonic*. A regular growth-rate function g is *payoff-monotonic* if, for all $\boldsymbol{z}$

$$\pi(\phi_i, \boldsymbol{z}) > \pi(\phi_j, \boldsymbol{z}) \quad \Leftrightarrow \quad g_i(\boldsymbol{z}) > g_j(\boldsymbol{z}) \tag{9}$$

payoff-monotonicity requires the growth-rates of each firm type to respect the internal ordering of their payoffs. The regularity of $\boldsymbol{g}$ ensures that the associated system of differential (difference) equations possesses a unique solution through any initial state in the unit simplex Δ, a solution that never leaves the simplex.

We are interested in identifying stable equilibrium population profiles. An *equilibrium* population profile $\boldsymbol{z}(t)$ is one where all strategies with $z_i(t) > 0$ earn the same payoff. This is a state of rest of the dynamic processes (7) and (8). However, as only *stable* equilibria are likely to occur, we use stability as a criterion to select between equilibria and rule out trivial cases. There are two concepts of stability: (i) *Lyapunov* stability and (ii) *asymptotic* stability. Broadly speaking, a population $\boldsymbol{z}(t)$ is Lyapunov stable if all solutions that start *sufficiently close* to $\boldsymbol{z}(t)$ stay *close*; a state $\boldsymbol{z}(t)$ is asymptotically stable if it is Lyapunov stable and if, in addition, any trajectories starting *sufficiently close* to $\boldsymbol{z}(t)$ approach $\boldsymbol{z}(t)$ as $t \to \infty$.

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[6] The population proportions are non-negative and add up to one in each period. Hence $\boldsymbol{z}(t) \in \Delta$, the unit simplex. The state vector $\boldsymbol{z}(t)$ is formally identical to a mixed strategy where the z_i represents the probability of playing pure-strategy ϕ_i.

[7] Or expected payoff in the case of the mixed strategy interpretation.

4. The Nash-equilibria of the conjecture game

As is well known, there is a close relationship between the dynamic equilibrium and the Nash-equilibrium. Thus, in order to understand the evolution of conjectures in the economy, we need to analyze further the stage-game $\Gamma = \{\Phi, \pi^i, i = a, b\}$. If we can identify the properties of the Nash-equilibrium, we will be able to identify and characterize the stable equilibria for the dynamics.

We interpret Γ as a hypothetical *conjecture game* between two firms. The game has two stages: in stage 1 the firms form their conjectures, and in the second stage the outputs are determined given the chosen conjectures (according to (4)). *It should be clear that the conjecture game is purely hypothetical, being a fiction which we use as modelers to understand the evolutionary dynamics of the economy*. The individual firms in our economy never play the conjecture game: in any period, they simply play the output game given their current conjectures.

4.1. Continuous strategies

In order to analyze the stage-game with a finite strategy set Φ, we will first analyze the related game where we treat conjectures ϕ^i as a continuous variable $\hat{\Gamma} = \{\phi^i \in [-1, 1], \pi^i, i = a, b\}$. There is a unique pure-strategy equilibrium; this is strict when $c > 0$ and weak if $c = 0$. Furthermore, the equilibrium conjectures are *consistent*.

Proposition 1. *Consider* $\hat{\Gamma} = \{\phi^i \in [-1, 1], \pi^i, i = a, b\}$.

(a) *Let* $c \geq 0$. *There exists a unique and symmetric Nash-equilibrium where both players have the consistent conjecture* ϕ^*.
(b) *If* $c > 0$, *the equilibrium is strict and* ϕ^* *is ESS*.
(c) *If* $c = 0$, *the equilibrium is weak and there exists no ESS*.

The conjecture game thus has a unique Nash-equilibrium in which both firms hold the consistent conjecture. In the case of strictly convex costs ($c > 0$), the Nash-equilibrium is strict and hence an *evolutionary stable strategy*[8] (*ESS*).

4.2. Finite strategy set Φ

We now turn to the case of a finite strategy set Φ, yielding the discrete conjecture game $\Gamma = \{\Phi, \pi^i, i = a, b\}$. As we move from the continuous case to the discrete case, the set of Nash-equilibria of the game changes. In particular whereas $\hat{\Gamma} = \{\phi^i \in [-1, 1], \pi_i, i = a, b\}$ is dominance solvable and yields a unique Nash-equilibrium, $\Gamma = \{\Phi, \pi^i, i = a, b\}$ need not be dominance solvable and the set of Nash-equilibria may contain more than one element. The analysis that follows identifies more precisely the equilibrium set and also gives an algorithm to compute the Nash-equilibria in Γ.

[8] For definition, see Hofbauer and Sigmund (1990).

A pure-strategy ϕ^i for player i is said to be *strictly dominated* by another pure-strategy $\hat{\phi}^i$ if for all ϕ^j, $\pi(\phi^i, \phi^j) < \pi(\hat{\phi}^i, \phi^j)$. A set of strategy pairs is denoted $\phi = (\phi^i, \phi^j) \in \Phi^2$, where Φ^2 is the product space of all possible strategy pairs. Consider a subset Ψ^τ of possible strategy pairs $\tilde{\phi}$, $\tilde{\phi} \in \Psi^\tau \subseteq \Phi$: we can define the set of player i's undominated responses to Ψ^τ:

$$U^i(\Psi^\tau) = \{\phi^i \in \Phi | (\forall \tilde{\phi}^i \in \Phi)(\exists \hat{\phi} \in \hat{\Psi}) | \pi(\phi^i, \hat{\phi}^j) \geq \pi(\tilde{\phi}^i, \hat{\phi}^j)\}.$$

That means if ϕ^i is in the set $U^i(\Psi^\tau)$, then for any $\tilde{\phi}^j$ occurring in Ψ^τ, ϕ^i yields as high a payoff as any other $\tilde{\phi}^i$ occurring in Ψ^τ.

Let $U(\Psi^\tau) = (U^i(\Psi^\tau), U^j(\Psi^\tau))$ be the list of undominated responses for each player, and let $\bar{U}(\Psi^\tau) \subseteq \mathfrak{R}$ denote the interval formed between the highest and the lowest undominated responses [*inf*($U(\Psi^\tau)$), *sup*($U(\Psi^\tau)$)]. We can then define a new subset $\Psi^{\tau+1}$ of strategy pairs, allowing only those pairs where both conjectures lie in the range $\bar{U}(\Psi^\tau)$. In moving from Ψ^τ to $\Psi^{\tau+1}$, we are in effect deleting those conjectures that were dominated.

Define $\Psi^0 = \Phi^2$, the full set of possible strategy profiles. A strategy ϕ^i is *serially undominated* if $\phi^i \in U^i(\Psi^\tau)$ for all τ. Let us denote the set of undominated strategies as $\Psi^u = \bigcap_{\tau=0}^{\infty} \Psi^\tau$, with conjectures in the range $[\underline{\phi}^u, \bar{\phi}^u]$. Conjectures in this set remain undeleted, never being strictly dominated.

We show that the set of pure-strategy Nash-equilibria $\mathcal{N}$ is non-empty, and that $\mathcal{N}$ contains the smallest and largest serially undominated strategies. Furthermore, for all serially undominated strategies ϕ' there is a symmetric equilibrium where both firms play the strategy.

Proposition 2. $\Gamma = \{\Phi, \pi^i, i = a, b\}$.

(a) *The set of pure-strategy Nash-equilibria of Γ is non-empty and contains the largest and smallest serially undominated strategies* $\{\underline{\phi}^u, \bar{\phi}^u\}$.
(b) *For any serially undominated strategy* $\phi' \in [\underline{\phi}^u, \bar{\phi}^u]$, *there exists a symmetric pure-strategy equilibrium* (ϕ', ϕ').

Finally, we need to relate the results directly to the concept of consistent conjectures. The consistent conjecture is always contained within the range of undeleted conjectures.

Proposition 3. *Consider the consistent conjecture* ϕ^*:

(a) *if* $\phi^* \in \Phi$, then $\phi^* \in \Psi^u$, *and* (ϕ^*, ϕ^*) *is a symmetric pure-strategy equilibrium and,*
(b) *if* $\phi^* \notin \Phi$, *then* $\phi^* \in [\underline{\phi}^u, \bar{\phi}^u]$.

Let us define the set of *almost consistent conjectures* $\Phi^* \subseteq \Phi$:

$$\Phi^* = \{\phi^i \in \Psi^u : \phi^i = \arg\min(\phi^* - \phi)^2\}. \tag{10}$$

In the case where $\phi^* \notin \Phi$, then Φ^* contains either one or both of the conjectures nearest to the consistent conjecture ϕ^*. Clearly, $\Phi^* \subseteq \Psi^u$. The almost consistent conjectures are undeleted, and hence there exists a symmetric pure-strategy equilibrium in which both firms have the same almost consistent conjecture.

Let us take an example of a discrete strategy set generated by forming a grid with maximum grid length $\delta \equiv \max_i |\phi^i - \phi^{i+1}|$. The next proposition considers what happens as δ becomes small. If δ is small enough, then undominated conjectures will be close to the consistent conjecture. Since consistent conjectures are negative, it implies that all undominated strategies become negative (i.e. more competitive than the Cournot conjecture).

Proposition 4. *Consider $\Gamma = \{\Phi, \pi^i, i = a, b\}$. Let $-\phi^* > \varepsilon > 0$. There exists a $\bar{\delta} > 0$ such that for all $\delta < \bar{\delta}$ if ϕ^i is a serially undominated strategy, then $|\phi^i - \phi^*| < \varepsilon$.*

5. The dynamic equilibrium

In this section we state the main result of the paper: the global convergence of payoff-monotonic dynamics to a small neighborhood of the consistent conjecture ϕ^*. We start from the implications of payoff-monotonic dynamics for aggregate equilibrium behavior, in particular its relationship with Nash-equilibrium in the stage-game $\Gamma = \{\Phi, \pi^i, i = a, b\}$ and serially dominated strategies. It has been shown that, (Nachbar, 1990; Bomze, 1986; Samuelson and Zhang, 1992)

(a) if $z \in \Delta$ is Lyapunov stable in (7), then z is a Nash-equilibrium of the stage-game;
(b) if $z \in \Delta$ is the limit to some interior solution to (7), then z is a Nash-equilibrium of the stage-game;
(c) if z is a strict Nash-equilibrium, then z is asymptotically stable; and
(d) if a pure-strategy does not survive the iterated elimination of pure strategies dominated by pure strategies, then its population share converges to zero in any payoff-monotonic dynamics from any initial state $z(0) \in \mathrm{int}(\Delta)$.

Lyapunov and asymptotic stability implies aggregate Nash behavior under the dynamics (7) and (8). Moreover, provided that the initial state $z(0)$ is completely mixed (i.e. has full support on Φ), all strategies outside the set of serially undominated strategies are eliminated by any payoff monotonic selection dynamics regardless of whether it converges. Cycles cannot be ruled out, but the strategies in the support must all be serially undominated.

What we now show is that if we can choose the grid to be fine enough, all surviving conjectures will be close to the consistent conjecture. To put it another way, the set of undominated strategies can be made arbitrarily close to the consistent conjecture, and there is global convergence to this set in the long-run.

Proposition 5. *Consider $\Gamma = \{\Phi, \pi^i, i = a, b\}$. Assume that $z(0)$ is completely mixed, and let $\delta \equiv \max_i |\phi^i - \phi^{i+1}|$ denote the maximum grid length. For arbitrarily small positive ϵ, there exists $\bar{\delta}(\epsilon) > 0$ such that for $\delta < \bar{\delta}(\epsilon)$,*

$$\text{If } \lim_{t\to\infty} z_i(t) > 0, \quad \text{then } |\phi_i - \phi^*| < \epsilon.$$

Whilst we know that there is convergence to the set of undominated strategies, there are different types of equilibria. We analyze the stability properties of the possible equilibria and show that pure strategy equilibria are in general asymptotically stable (because they are strict equilibria) whilst equilibrium states that mimic mixed strategy equilibria are unstable due to not satisfying Lyapunov stability conditions.

Proposition 6. *Consider* $\Gamma = \{\Phi, \pi^i, i = a, b\}$.

(a) *pure-strategy Nash-equilibria* $(\phi^i, \phi^j) \in \mathcal{N}$ *are generally strict and therefore are asymptotically stable*.
(b) *All equilibrium states* z^* *in which* $\#C(z^*) > 1$ *are Lyapunov unstable*.

If more than one strategy is played in equilibrium, a perturbation to z^*, however small, leads the system to a different equilibrium where only one of the two strategies survives. It should be clear that, if present in Φ, the consistent conjecture is an element of the set of serially undominated strategies. However, due to the discreteness of the strategy space, there might be more than one undominated strategy. However, as we know from Proposition 4, if the distance between consecutive strategies in Φ is made small enough, then the set of serially undominated strategies can be contained in an arbitrarily small neighborhood of the consistent conjecture. Furthermore, Proposition 6 tells us that only the pure strategy equilibria where all firms have the same conjecture are stable.

6. Conclusions

In this work, we have taken a model in which firm behavior depends on firm beliefs. Firms play the conjectural variation duopoly, and their belief is their conjecture about the slope of other firms' reaction functions. We analyze this in the context of an evolutionary framework, in which more successful types of rules become more common. We model the Darwinian process using payoff-monotonic selection dynamics. Analytically, we find that in the case of strictly convex costs, the unique *ESS* is the consistent conjecture. In the case of no costs, there exists no *ESS*, and the consistent conjecture (Bertrand) is a dominated type.

We consider the case of a finite set of possible conjectures and study the properties and equilibria of the dynamics. Analytically, we find that for a fine enough grid, the dynamics converge to a small neighborhood of the consistent conjecture. This set is formed by the serially undominated strategies, and we show that all symmetric serially undominated strategy profiles are pure-strategy Nash equilibria. These symmetric equilibria are generally strict and therefore asymptotically stable in the class of dynamics considered. The serially undominated strategies (conjectures) are negative, and the equilibrium is therefore more competitive than the Cournot equilibrium, confirming similar findings of other papers dealing with dynamic duopoly games (Fershtman and Kamien, 1987; Reynolds, 1987; Maskin and Tirole, 1987; Driskill and McCafferty, 1989).

We believe that the framework proposed in this model provides a rationale for consistency of conjectures based on bounded rationality and evolutionary selection. Most of the criticisms of the consistency and conjectural variations have focused on the issue from the perspective of classical game theory based on perfect rationality and common knowledge. Whilst we accept these criticisms as valid within their own framework, our approach is based on different, and we believe more plausible, foundations.

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Appendix A. Proofs

A.1. Proposition 1

(a) Maximization of the payoffs functions as given in (5) with respect to ϕ^i yields the best response function β:

$$\phi^i = \beta\left(\phi^j\right) = -\frac{1}{2+c+\phi^j}, \quad i, j = a, b, \quad i \not\equiv j. \tag{A.1}$$

The best response conjecture corresponds to the actual slope of the rival's reaction function in output space. Solving the above system yields the unique Nash-equilibrium conjectures for the two firms. As the algebra shows, both firms have the same conjecture, and it is equal to the consistent conjecture ϕ^* from Section 2:

$$\phi^i = \phi^j = \phi^* = -\left(1+\frac{c}{2}\right) + \sqrt{\frac{4c+c^2}{4}}. \tag{A.2}$$

Note that $0 > \phi^* \geq -1$, with $|\phi^*|$ strictly decreasing in c. Uniqueness is established noting that $\beta(\cdot)$ is a contraction mapping. Since

$$0 < \frac{\mathrm{d}\beta(\cdot)}{\mathrm{d}\phi^j} = \frac{1}{(2+c+\phi^j)^2} < 1 \tag{A.3}$$

for $\phi \in [-1, k]$ (where $k \geq 1$), and $c \geq 0$, this implies that the solution is unique.

(b) If $c > 0$, then the payoff function is *strictly* concave in ϕ_i, hence the equilibrium is strict and *ESS*.

(c) If $c = 0$, the equilibrium conjecture is equal to -1 and implies zero profits whatever the conjecture of the rival firm so that the equilibrium is weak. From the definition of ESS it must be that $\pi(-1, -1) \geq \pi(\phi^i, -1)$ and if $\pi(-1, -1) = \pi(\phi^i, -1)$ then, $\pi(-1, \phi^j) > \pi(\phi^i, \phi^j)$ noting that $\pi(-1, \phi^j) = \pi(\phi^i, -1) = 0, \forall \phi^i, \phi^j$ and that $\pi(\phi^i, \phi^j) > 0, \forall \phi^i, \phi^j \not\equiv -1$, the only Nash-equilibrium of the game, given by $(-1, -1)$ is not ESS. Moreover being the equilibrium unique, there are no others ESS.

A.2. Proposition 2

A.2.1. Part (a)

Consider the full set of strategy profiles Ψ^0 and let $\bar{\phi}^0 = \sup(\Psi^0)$ and $\underline{\phi}^0 = \inf(\Psi^0)$. For $k \geq 1$ define $\bar{\phi}^k = \overline{\mathrm{BR}}(\bar{\phi}^{k-1})$ and $\underline{\phi}^k = \underline{\mathrm{BR}}(\underline{\phi}^{k-1})$. Where $\overline{\mathrm{BR}}(\bar{\phi}^{k-1})$ and $\underline{\mathrm{BR}}(\underline{\phi}^{k-1})$ denote the collections of the largest and smallest best replies for players i and j to $\bar{\phi}^{k-1}$ and $\underline{\phi}^{k-1}$, respectively. Firstly, we show that $U(\Psi^k) \subset [\underline{\phi}^k, \bar{\phi}^k]$. This is trivially true for $k = 0$; assume it is true for $k < r$. Then

$$U(\Psi^{r+1}) \subset U([\underline{\phi}^r, \bar{\phi}^r]) \subset [\underline{\phi}^{r+1}, \bar{\phi}^{r+1}] \tag{A.4}$$

The first inclusion follows from the observation that U is a monotonic non-decreasing function; the second inclusion is readily proved using two properties of the payoff function:

P1 $$\frac{\delta\pi(\phi^i, \phi^j)}{\delta\phi^i} = \begin{cases} > 0 & : \quad \phi^i < \hat{\phi}^i \\ < 0 & : \quad \phi^i > \hat{\phi}^i \end{cases};$$

P2 Monotonicity of the best reply: If ϕ^i is a best response to $\phi^{j'}$ and $\hat{\phi}^i$ is a best response to $\tilde{\phi}^j$, where $\tilde{\phi}^j > \phi^{j'}$, then $\hat{\phi}^i \geq \phi^i$.

where $\hat{\phi}^i$ is the best response to ϕ^j. By definition $\underline{\mathrm{BR}}(\underline{\phi}^k)$ and $\overline{\mathrm{BR}}(\bar{\phi}^k)$ are in $U([\underline{\phi}^k, \bar{\phi}^k])$ and thus $[\underline{\mathrm{BR}}(\underline{\phi}^k), \overline{\mathrm{BR}}(\bar{\phi}^k)] \subset \bar{U}([\underline{\phi}^k, \bar{\phi}^k])$. Consider a strategy profile $\tilde{\phi} \notin [\underline{\mathrm{BR}}(\underline{\phi}^k), \overline{\mathrm{BR}}(\bar{\phi}^k)]$ and let $\tilde{\phi}^i > \phi^{*i} \equiv \overline{\mathrm{BR}}^i(\bar{\phi}^k)$. We claim that $\tilde{\phi}^i \notin U[\underline{\phi}^k, \bar{\phi}^k]$ because ϕ^{*i} strictly dominates $\tilde{\phi}^i$. Indeed for any $\phi \in [\underline{\phi}^k, \bar{\phi}^k]$,

$$\pi(\tilde{\phi}^i, \phi^j) - \pi(\phi^{*i}, \phi^j) < 0 \tag{A.5}$$

because ϕ^{*i} is the largest best reply to the largest profile $\bar{\phi}^k$, the best reply to any other profile in $[\underline{\phi}^k, \bar{\phi}^k]$ is smaller than ϕ^{*i} thanks to P2. In addition property P1 ensures that the above inequality holds. Similarly if $\phi^{*i} < \underline{\mathrm{BR}}(\underline{\phi}^k)$. Hence $\sup U([\underline{\phi}^k, \bar{\phi}^k]) = \overline{\mathrm{BR}}(\bar{\phi}^k)$ and $\inf U([\underline{\phi}^k, \bar{\phi}^k]) = \underline{\mathrm{BR}}(\underline{\phi}^k)$, and we conclude that $\bar{U}([\underline{\phi}^k, \bar{\phi}^k]) = [\underline{\mathrm{BR}}(\underline{\phi}^k), \overline{\mathrm{BR}}(\bar{\phi}^k)]$.

The sequences $\bar{\phi}^0, \bar{\phi}^1, \dots, \bar{\phi}^m$ and $\underline{\phi}^0, \underline{\phi}^1, \dots, \underline{\phi}^m$ have limits $\bar{\phi}$ and $\underline{\phi}$, respectively. We now show that $\bar{\phi}$ (and similarly $\underline{\phi}$) is a Nash equilibrium profile. If this were not the case, then for some player $i = a, b$ there exists some other $\hat{\phi} \in [\underline{\phi}, \bar{\phi}]$ such that

$$\pi(\hat{\phi}^i, \bar{\phi}^j) > \pi(\bar{\phi}^i, \bar{\phi}^j), \tag{A.6}$$

but then again the monotonicity of best replies coupled with P1 implies that $\hat{\phi}$ strictly dominates $\bar{\phi}$, which contradicts $\bar{\phi}$ being the limit of the sequence.

A.2.2. Part (b)

Having proved part (a) of the proposition, we only have to show that the set of Nash-equilibria is a closed, bounded and convex subset of Φ. Let $\delta = |\hat{\phi} - \phi|$ be the distance between any two adjacent strategies in Φ. Define $\bar{V}^i(\phi, \delta) = \pi(\phi^i - \delta, \phi^j) - \pi(\phi^i, \phi^j)$, where $\pi(\phi^i - \delta, \phi^j)$ is the payoff to player i adopting the strategy immediately to the left of the strategy adopted by player j (ϕ^j). Similarly let $\underline{V}^i(\phi, \delta) = \pi(\phi^i + \delta, \phi^j) - \pi(\phi^i, \phi^j)$. Define

$$\bar{S}^i = \{\phi^i \in \Phi | \bar{V}^i(\phi, \delta) \leq 0\} \tag{A.7}$$

and

$$\underline{S}^i = \{\phi^i \in \Phi | \underline{V}^i(\phi, \delta) \leq 0\}. \tag{A.8}$$

P1 implies that if

$$\pi(\hat{\phi}^i, \phi^j) < \pi(\phi^i, \phi^j), \quad \text{for } \hat{\phi} < \phi, \tag{A.9}$$

then

$$\pi(\tilde{\phi}^i, \phi^j) < \pi(\phi^i, \phi^j), \quad \forall \tilde{\phi} < \hat{\phi}$$

and similarly for $\hat{\phi} > \phi$, $\forall \tilde{\phi} > \hat{\phi}$. For a given δ, both $\underline{V}^i(\phi, \delta)$ and $\bar{V}^i(\phi, \delta)$ are continuous in ϕ and cross the horizontal axis only once over the range $[-1, k]$, $k \geq 1$. It follows that $\underline{S}^i$ and $\bar{S}^i$ are closed, bounded and convex subsets of Φ.

Define the intersection of $\underline{S}^i$ and $\bar{S}^i$ as $S^i = \underline{S}^i \cap \bar{S}^i$ and let $\mathcal{E} = \{(\phi^i, \phi^j) \in S = S^i \times S^j | \phi^i = \phi^j\}$ be the set of symmetric strategy profiles in S. We now show that ϕ is a Nash profile *iff* $\phi \in \mathcal{E}$. The *only if* part of the statement is obvious from the definition of Nash-equilibrium. The *if* part is proved by contradiction. Consider a profile $\tilde{\phi} \in \mathcal{E}$ and assume that it is not a Nash equilibrium; this implies that there exists a strategy ϕ^i such that $\pi(\phi^i, \tilde{\phi}^j) > \pi(\tilde{\phi}^i, \tilde{\phi}^j)$. Assume that $\phi > \tilde{\phi}$; it follows that $\pi(\tilde{\phi}^i + \delta, \tilde{\phi}^j) > \pi(\tilde{\phi}^i, \tilde{\phi}^j)$ and $\underline{V}^i(\tilde{\phi}, \delta) > 0$ contradicting the assumption that $\tilde{\phi} \in \mathcal{E}$. *Mutatis mutandis*, the same contradiction is derived assuming $\phi < \tilde{\phi}$.

A.3. Proposition 3

Part (a) This is trivial; ϕ^* is the unique serially undominated strategy in $\hat{\Gamma}$, and it remains undominated if present in Φ.

Part (b) We prove it by contradiction. Clearly $\underline{\phi}^u$ and $\bar{\phi}^u$ are, respectively, the smallest $\phi \in \Phi$ such that $\underline{V}^i(\phi, \delta) \leq 0$ and the largest $\underline{V}^i(\phi, \delta) \leq 0$. We also know that $\bar{\phi}^u \geq \underline{\phi}^u$. Consider any finite strategy set Φ spanning the interval $[-1, 1]$ and assume that $\phi^* \notin [\underline{\phi}^u, \bar{\phi}^u]$. This implies that either $\underline{\phi}^u > \phi^*$ or $\bar{\phi}^u < \phi^*$. Suppose that $\underline{\phi}^u > \phi^*$; this will imply that $\pi(\phi^* + \delta, \phi^*) < \pi(\phi^*, \phi^*)$ which contradicts that ϕ^* is a best reply to itself. Similar results occur if we assume that $\bar{\phi}^u < \phi^*$.

A.4. Proposition 4

Let us assume the contrary: there exists some $\phi^* < \hat{\phi} < 0$ such that for all $\delta > 0$, $\bar{\phi}^u > \hat{\phi}$ (and analogously for undominated conjectures below ϕ^*). In this case, since $\bar{\phi}^u$ is undominated for all $\delta > 0$,

$$\Pi(\bar{\phi}^u - \delta, \bar{\phi}^u) \leq \Pi(\bar{\phi}^u, \bar{\phi}^u). \tag{A.10}$$

But, for any $\bar{\phi}^u > \phi^*$, $\beta(\bar{\phi}^u) < \bar{\phi}^u$, with

$$\Pi(\beta(\bar{\phi}^u), \bar{\phi}^u) > \Pi(\bar{\phi}^u, \bar{\phi}^u). \tag{A.11}$$

Let $\bar{\delta} = \hat{\phi} - \beta(\hat{\phi})$. Then for $\delta < \bar{\delta}$,

$$\Pi((\bar{\phi}^u - \delta), \bar{\phi}^u) > \Pi(\bar{\phi}^u, \bar{\phi}^u), \tag{A.12}$$

the desired contradiction. The same argument holds when there is a lower bound $\check{\phi} < \phi^*$ such that $\underline{\phi}^u < \check{\phi}$ for all $\delta > 0$. Hence as $\delta \to 0$, $\underline{\phi}^u \to \phi^*$ and $\bar{\phi}^u \to \phi^*$, from which the Proposition immediately follows.

A.5. *Proposition 5*

By Proposition 4, all undominated strategies can be made arbitrarily close to the consistent conjecture ϕ^* by choosing a maximum grid length small enough. Payoff-monotonic dynamics eliminate serially dominated strategies (Samuelson and Zhang); thus, given completely mixed initial conditions, the only surviving strategies in the long-run belong to the set of undominated strategies Ψ^u. It follows immediately that if δ is small, any surviving strategy will be close to ϕ^*.

A.6. *Proposition 6*

A.6.1. *Part (a)*

All ϕ such that $\underline{\phi}^u < \phi < \bar{\phi}^u$ are strict Nash-equilibria because both $\underline{V}(\phi, \delta)$ and $\bar{V}(\phi, \delta)$ are negative. All strict Nash-equilibria are asymptotically stable under the dynamics considered, and $\underline{\phi}^u$ and $\bar{\phi}^u$ are in general strict Nash-equilibria, being weak only if by chance $\underline{V}(\underline{\phi}^u, \delta) = 0$ or $\bar{V}(\bar{\phi}^u, \delta) = 0$. However, the case of a weak equilibrium is not robust. A small perturbation to δ, the distance between $\bar{\phi}^u$ ($\underline{\phi}^u$) and the strategy immediately to its left (right), will not alter the set $\mathcal{N}$ of Nash-equilibria and will make $\underline{\phi}^u$ ($\bar{\phi}^u$) a strict equilibrium.

A.6.2. *Part (b)*

We show that such states z^* do not satisfy the conditions for Lyapunov stability. Taking the case where $\#C(z) = 2$, with $\phi_i < \phi_j$, the profit to strategies ϕ_i and ϕ_j are, respectively

$$\bar{\pi}_i = z_i \pi_{i,i} + (1 - z_i)\pi_{i,j}, \tag{A.13}$$

and

$$\bar{\pi}_j = z_i \pi_{j,i} + (1 - z_i)\pi_{j,j}. \tag{A.14}$$

Recall that these strategies belong to the set of serially undominated strategies and therefore are strict best replies to themselves:

$$\pi_{j,i} < \pi_{i,i} < \pi_{i,j} < \pi_{j,j}$$

and $(\mathrm{d}(\bar{\pi}_i - \bar{\pi}_j))/(\mathrm{d}z_i) > 0$. Given a neighborhood V of z^*, a trajectory originating from it will converge to a monomorphic equilibrium state where only one of the two strategies survives. The proof easily generalizes to the case where there are more than two strategies in the support $C(z^*)$.

References

Bomze, I., 1986. Non-cooperative two-person games in biology: a classification. International Journal of Game Theory 15, 31–57.

Boyer, M., Moreaux, M., 1983. Consistent versus non-consistent conjectures in Duopoly theory: some examples. Journal of Industrial Economics 32, 97–110.

Bresnahan, T.F., 1981. Duopoly models with consistent conjectures. American Economic Review 71, 934–945.

Cabral, L.M.B., 1995. Conjectural variations as a reduced form. Economic Letters 49, 397–402.

Dockner, E.J., 1992. A dynamic theory of conjectural variations. Journal of Industrial Economics 40, 377–395.

Driskill, R.A., McCafferty, S., 1989. Dynamic duopoly with adjustment costs: a differential game approach. Journal of Economic Theory 49, 324–338.

Fershtman, C., Kamien, M.I., 1987. Dynamic duopolistic competition with sticky prices. Econometrica 55, 1151–1164.

Hahn, F., 1977. Exercises in conjectural equilibrium. In: Strom, S., Werin, L. (Eds.), Topics in Disequilibrium Economics. McMillan, London, pp. 64–80.

Hahn, F., 1978. On non-Walrasian equilibrium. Review of Economic Studies 45, 1–17.

Hofbauer, J., Sigmund, K., 1990. Adaptive dynamics and evolutionary stability. Appl. Math. Lett. 3, 75–79.

Kamien, M.I., Schwartz, N.L., 1983. Conjectural variations. Canadian Journal of Economics 16, 191–211.

Lindh, T., 1992. The inconsistency of consistent conjectures. Journal of Economic Behavior and Organization 18, 69–80.

Makowsky, L., 1987. Are rational conjectures rational? Journal of Industrial Economics 36, 35–47.

Martin, S., 1993. Advanced Industrial Economics. Blackwell Scientific Publications, Oxford.

Maskin, E., Tirole, J., 1987. A theory of dynamic oligopoly, III: Cournot competition. European Economic Review 31, 947–968.

Nachbar, J., 1990. Evolutionary selection dynamics in games: convergence and limit properties. International Journal of Game Theory 19, 59–89.

Perry, M.K., 1982. Oligopoly and consistent conjectural variations. Bell Journal of Economics 13, 197–205.

Reynolds, S.S., 1987. Capacity investment, preemption and commitment in an infinite Horizon model. International Economic Review 28, 69–88.

Samuelson, L., Zhang, J., 1992. Evolutionary stability in asymmetric games. Journal of Economic Theory 57, 363–391.

Shapiro, C., 1989. Theories of oligopoly behaviour. Schmalensee, R., Willig, R. (Eds.), Handbook of Industrial Organization. North-Holland, Amsterdam, pp. 329–414.

Ulph, D., 1983. Rational conjectures in the theory of oligopoly. International Journal of Industrial Organization 1, 131–154.