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Endogenous fluctuations in an open economy with increasing returns to scale

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Abstract

In this paper we study the effects of opening an economy, with increasing returns in the production of nontraded goods, on the existence of multiple Pareto-ranked stationary equilibria, local indeterminacy and bifurcations. We consider a standard *overlapping generation model* of a small open economy, with a fixed exchange rate, where labour is the only input and money the only asset. We find that when there are increasing returns, the open economy may display persistent equilibrium endogenous fluctuations (deterministic and stochastic) in the balance of trade and main macroeconomic aggregates. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

This paper develops a simple model of a small open economy showing how persistent fluctuations in the balance of trade and main macroeconomic aggregates can arise into the system without appealing to shocks to the fundamentals.

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In particular, we study the impact of openness on the uniqueness/multiplicity of steady states, local indeterminacy, deterministic and stochastic endogenous fluctuations in an economy with increasing returns to scale and imperfect competition.

We consider a simple OG open economy model, with money, in which there is a traded and a nontraded sector. The exchange rate is assumed fixed, the price of traded goods being determined by the world market. For simplicity, we assume that the traded good is an endowment which is constant over time. The technology for producing the nontraded good with labour exhibits increasing returns to scale. There is imperfect (Cournotian-monopolistic) competition in the nontraded sector output markets with free entry. In our set up the (endogenous) equilibrium number of firms is constant and so is the mark up. The model is designed to be as simple as possible, and we abstract away from capital accumulation. In this economy, money is the only asset and is predetermined (its value at time t is determined by the state of the economy in the previous period, i.e. by the balance of trade). Technology and preferences are time invariant, as is the price of traded goods (exogenously fixed, due to the small country and fixed exchange rate assumptions).We disregard exogenous shocks in order to focus on the role played by underlying preference and technology parameters in generating endogenous fluctuations.

The present paper develops the existing studies of endogenous fluctuations in the macroeconomic context to the case of an open economy. The earliest studies followed Grandmont (1985) in focussing on the closed economy case. We follow his approach by assuming that there is no capital: however, we differ from him in assuming that there are increasing returns in production, and also that there are two sectors (traded and nontraded) so that the dynamic system is two dimensional. This complements the closed economy models with capital, in which endogenous fluctuations are possible with positive labour supply elasticities (for the case of constant returns, see Reichlin (1986, 1992), Woodford (1986) and Grandmont et al. (1998)) or, for the case of increasing returns, recent works by Lloyd-Braga (1995a,b) and Cazzavillan et al. (1998).

In our study we explore the conditions under which local deterministic and stochastic endogenous fluctuations may emerge into the system by using the methodology of Grandmont et al. (1998). The advantage of this method is that it provides a simple and comprehensive geometrical way to analyse the behaviour of equilibrium trajectories nearby a steady state, without appealing to particular specifications of preferences or technology. We find that there are four fundamental parameters determining the local dynamic properties of the model: the extent of returns to scale; the degree of substitutability between traded and nontraded goods; the propensity to consume nontraded goods; the elasticity of labour supply. Our main results are the following.

First, we characterise sufficient conditions for the existence of a steady-state solution (Proposition 1) and the existence of more than one steady state

(Propositions 2 and 3). We find that the equilibria are Pareto ranked (Proposition 4): steady states with higher levels of nontraded consumption Pareto dominate those with lower levels. A constant elasticity formulation of preferences and technology illustrates Propositions 1–4.

Second, we analyse the local dynamics of the system and consider local stability and the emergence of local bifurcations. We find that (Proposition 5), whilst for some parameter values the economy is saddle point stable, for other parameter values local bifurcations (*transcritical* or *flip*) can occur and local (deterministic and stochastic) endogenous fluctuations emerge. In addition, stochastic endogenous fluctuations can also occur when the nearby steady state is a sink (i.e. locally indeterminate) or close to a stable periodic cycle (i.e. supercritical flip bifurcation).

In general, endogenous fluctuations are more likely if returns to scale are larger and the economy is more open. Imperfect competition is important but not crucial in our model: (internal) increasing returns to scale in production are only possible if there is imperfect competition. However, if increasing returns are external, then much the same results would hold under perfect competition. Thus it is the degree of increasing returns (internal or external) that is the most fundamental requirement: if there are constant or decreasing returns to scale, then endogenous fluctuations are not possible.[1] Openness is crucial for the emergence of deterministic cycles. In the closed economy version of our simple model (see Lloyd-Braga, 1995a), flip bifurcations cannot occur with a positively sloped labour supply even with increasing returns to scale.

Our paper departs from the existing intertemporal open macroeconomic literature. One of the major puzzles in the intertemporal approach to the current account has been its excess volatility (Baxter, 1995). In particular, if one views supply shocks as being the driving force behind macroeconomic fluctuations, these need to be persistent shocks if they are to explain output and employment. Persistent supply shocks should lead to persistent fluctuations on the current account (see Obstfeld and Rogoff (1996) for a survey), but trade and current accounts appear to be more volatile than predicted by this approach. Our approach, in contrast, explains the possibility that the volatility of the current

[1] The importance of increasing returns relates nicely to its increasing use in open economy RBC literature (e.g. Obstfeld and Rogoff, 1995; Beaudry and Devereux, 1995) although the context is rather different. As is common practice in the RBC literature, these papers only focus on saddle point stable equilibrium paths. They both analyse two-country models with capital accumulation and look at the effect of temporary and permanent monetary shocks on main macro aggregates.

account and related macro aggregates is not the result of exogenous shocks, but rather the result of endogenous fluctuations.[2]

The plan of the paper is as follows. In Section 2 we present the basic macroeconomic model. In Section 3 we consider the equilibrium dynamic system and explore its steady-state properties. In Section 4 we undertake the technical analysis of local dynamics and bifurcations around a steady state, seeing the relationship between the underlying parameters and the type of endogenous fluctuations that can emerge. In Section 5 we discuss the economic intuition of the results of Section 4. In Section 6 we conclude.

2. The model

In this paper we consider an OG model for a small open economy over an infinite sequence of discrete time periods $t = 1,2,\ldots,\infty$. There are two composite commodities: one is a nontraded good c which is produced and consumed domestically; the other is an internationally traded good x that is produced and consumed domestically and abroad (we follow convention and aggregate over goods which are net imports and those that are net exports). We assume that c is a composite good of a large fixed number of different goods, defined by a utility index of Dixit–Stiglitz type. Each good c_i is produced by n firms under Cournotian-monopolistic competition,[3] with free entry and exit. Firms have identical technology exhibiting increasing returns to scale (due to decreasing marginal cost).

Money is the numeraire and is the only asset (there is assumed to be no government in this economy). The exchange rate is fixed and normalized to unity. The price of the traded good, p^*, is then fixed (small country assumption).

Population is constant over time and is composed by a finite number of households living for two periods and acting under perfect foresight. In each period there are h young households and h old households. When young, households work, receive profits and save money; when old, they spend money and consume. Households have identical preferences described by the separable utility function: $U(c_{t+1}, x_{t+1}) - V(N_t/B)$, where $B > 0$ is a scaling parameter and N_t is the labour supply. We assume the following:

[2] One frequent objection to OG models with two-period lived agents is that fluctuations occur on time scales long compared to the lifetime of agents. However, in the paper we show that, in our set up, it is possible to reinterpret the household's behaviour as coming out of infinitely long lived agents consuming and working in every period and facing capital market imperfections, as in Woodford (1986). Therefore, the periods' length can be made compatible with business cycle frequencies.

[3] See d'Aspremont (1995). Producers in each market i behave à la Cournot, taking as given the prices of the other markets.

Assumption 1. The function $U(c,x)$ is a continuous concave function, homogenous of degree one for all $c,x \geq 0$, and C^r for $c,x > 0$ with r large enough. Moreover, $U'_c, U'_x > 0$ for $c,x > 0$ and $\lim_{c/x \to 0} U'_c/U'_x = \pm\infty$, $\lim_{c/x \to +\infty} U'_c/U'_x = 0$. $V(N_t)$ is a continuous function for $0 \leq N \leq N^*$, where N^* is the workers endowment of labour (possibly infinite); and C^r for $0 < N < N^*$ with r large enough. It also satisfies $V', V'' > 0$ and $\lim_{N \to N^*} V''(N) = +\infty$.

From now on we drop the time subscript whenever this does not give rise to any ambiguity.

2.1. Producers

Looking more closely at the nontraded sector, there are many markets $i = 1, \ldots, s$, each one with n firms $k = 1, \ldots, n$. Firms produce under the same technology, given by

$$y = N^r, \quad r \geq 1 \tag{1}$$

where y and N are, respectively, the output and labour input per firm in each period.

Each household has a CES subutility function over the outputs of the nontraded sector, with elasticity of substitution $1/\mu > 1$, i.e.:

$$c = \left(\frac{1}{s}\right)^{\mu/(1-\mu)} \left[\sum_{1}^{s} c_i^{1-\mu}\right]^{1/(1-\mu)} \tag{2}$$

yielding the inverse demand curve:

$$p_i = (c_i)^{-\mu}\, p\, (c/s)^{\mu} \tag{3}$$

where p_i is the price of each nontraded good and p is the aggregate price index for c, which is given by

$$p = \left(s^{-1} \sum_{1}^{s} p_i^{(\mu-1)/\mu}\right)^{\mu/(\mu-1)}. \tag{4}$$

Firms in each sector i decide the quantity to produce taking into account the effects of its own decisions at the sector level, but consider negligible any possible effect on output and price at the economy level. The labour market is perfectly competitive and firms take the wage rate (w) as given. A firm k in sector i solves the following problem:

$$\underset{N_{ik} \in \mathbf{R}_{++}}{\mathrm{MAX}} \left(p_i N_{ik}^r - w N_{ik}\right)$$

$$\text{s.t.} \quad \text{Eq. (3)}, \quad hc_i = \sum_j \bar{N}_{ij}^r + N_{ik}^r; \quad j = 1, \ldots, n, \quad j \neq k. \tag{5}$$

At the symmetric Cournot–Nash equilibrium[4] within sector i, the first-order condition can be written as follows:

$$p_i\left(\frac{n-\mu}{n}\right)=\frac{w}{r}N_{ik}^{1-r}. \tag{6}$$

Eq. (6) gives the condition for each firm to be on its reaction function within sector i in a symmetric equilibrium. The number of firms is determined by the zero profit condition (given our free entry assumption), i.e. the own-product wage equals the average product of labour:

$$w/p_i=N_{ik}^{r-1}. \tag{7}$$

If we combine the two equations above, we obtain the equilibrium number of firms with free entry: $n=\mu r/(r-1)$. This result says that the equilibrium number of firms in each sector is determined by the taste and technology parameters (μ, r), independently of the rest of the model.[5] This feature is taken from Lloyd-Braga (1995a) with the result that aggregate output varies in proportion to output per firm in each industry. Moreover, the mark up is constant and equals $r \geq 1$. Note that if $r=1$ an infinite number of firms would enter the market and the market power would vanish. Perfect competition is then the limiting case of constant returns to scale.

Since firms are identical and every sector i faces identical demand, the equilibrium is symmetric across sectors. Therefore, $N_{ik}=N$ and $p_i=p$. To simplify the exposition we assume that the total number of firms equals the number of households in each generation: $h=sn$. Using Eqs. (2), (5) and (7) the following equilibrium conditions must be satisfied in each period t:

$$c_t=N_t^r, \tag{8}$$

$$w_tN_t=p_tN_t^r. \tag{9}$$

Eq. (8) defines the equilibrium in the nontraded good sector while Eq. (9) determines the labour demand curve with an elasticity $\varepsilon^d=1/(r-1)$.

Before analysing the traded sector, it is important to stress that the above equilibrium conditions apply also to the case of a perfectly competitive economy with externalities. This analogy holds as long as: (i) the degree of social increasing

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[4] Using similar procedures as in d'Aspremont (1995) it can be shown that second-order conditions are satisfied and that any Cournot equilibrium must be symmetric in quantities across the active firms.

[5] There are restrictions on the feasible pairs of (μ, r) given by the constraint that $n \geq 2$: in particular, if $\mu < 2$, then $r \leq 2/(2-\mu)$. For the case of $\mu = 1$, we get the restriction $r \leq 2$ as in Lloyd-Braga (1995a).

returns, $(r-1)$, is constant and identical in both set ups; (ii) profits are zero and the mark up is constant when increasing returns are internal to the firm.[6] This implies, in turn, that our analysis of endogenous fluctuations in a small open economy can be extended to a wider range of market structures.

As regard to the modelling of the traded sector, we assume, for simplicity, that the home production of the traded good is constant through time and exogenously fixed at a level $\bar{x}>0$. The fixed output can be viewed as due to a natural resource or capacity constraint, or reflecting some 'Natural rate' of output in that sector, unaffected by the rest of the economy.[7] The revenue from the traded sector in every period t is $p^*\bar{x}$, which for simplicity we assume takes the form of profits. Whilst we have disaggregated the nontraded sector we will not disaggregate the traded sector. Since we are not modelling it as imperfectly competitive, and price and quantity produced are both exogenous here, we can treat it as a single sector representing some composite good.

2.2. Households

An household born at $t \geq 0$ can be seen as solving:

$$\underset{(c_{t+1},x_{t+1},N_t,M_t)\in R^4{}_{++}}{\text{MAX}} \quad (U(c_{t+1},x_{t+1})-V(N_t/B))$$

$$\text{s.t. } M_t = w_t N_t + p^*\bar{x}, \tag{10}$$

$$p^* x_{t+1} + p_{t+1} c_{t+1} = M_t, \tag{11}$$

[6] Suppose that the nontraded sector is affected by a productive externality (as in e.g. Benhabib and Farmer (1994) or Cazzavillan et al. (1998)) such that labour marginal productivity is positively affected by an increase in the aggregate labour stock; i.e. $y_{ik} = N^{v} N_{ik}$, where N^{v} is the externality and N is the average level of employment. Since N^{v} (as an externality) is taken as given by each firm, the marginal productivity of labour is $w/p = N^{v}$. From direct inspection of Eq. (9), one can easily see that these expressions are identical if we assume $v = r - 1$.

[7] There are a variety of ways of modelling households in a two-sector open economy. See Obstfeld and Rogoff (1996), (Section 10.2) for an approach similar to ours. Since we are interested in the intertemporal features of the model, we have opted for the most simple intersectoral structure: we have not allowed for variable production in the traded sector (as in Dixon (1994)). Note that the fixed output assumption can be reinterpreted in another way, i.e. as the result of a model with sector specific workers and no labour mobility across sectors. In this case the output of the traded sector will be constant, so long as: (i) workers in the traded sector have a constant marginal disutility of labour; (ii) the marginal productivity of labour in the traded sector is constant and higher than the reservation wage (measured in units of traded goods); (iii) there is perfect competition in this sector. In that case, revenues from the traded sector take the form of wages. Assuming that all households have the same utility $U(c, x)$, the distribution of income will be irrelevant for the equilibrium outcome and the same equilibrium conditions apply.

where M_t is the (per capita) stock of money at the outset of period $t+1$ (or end of period t). The constraints (10) and (11) show that households save through money holdings the income received while young to be spent in consumption goods while old. Note that using Eqs. (8)–(11) we obtain the asset accumulation equation:

$$M_t - M_{t-1} = p^*(\bar{x} - x_t).$$

The equation above shows, as expected, that in an economy with no government and fixed exchange rates (where money is the unique asset) the balance of trade determines the dynamics of the money stock.

It should be also emphasised that, in this setting, it is always possible to reinterpret the household's behaviour, at least nearby the steady state, as coming out of infinitely long-lived agents consuming and working in every period, but subject to cash-in-advance constraints, as in Woodford (1986).[8]

The equilibrium conditions for the household are obtained as follows. Since $U(c, x)$ is homogeneous of degree one, we can always write

$$U(c, x) = x.u(a), \quad \text{with } a \equiv c/x.$$

We can then define the marginal utility of the nontraded and traded good respectively, as follows:

$$g(a) \equiv u'(a), \quad q(a) \equiv u(a) - a.u'(a).$$

[8] As an infinitely lived agent our representative household maximises:

$$\sum_{t=1}^{\infty} \left[\beta^{t-1} U(c_t, x_t) - \beta^t V(N_t/B) \right]$$

where $0 < \beta < 1$ is the discount factor and the functions U and V satisfy the same assumptions as in our OG model. The budget constraints she/he has to face, in period t, are

$$M_t + p^* x_t + p_t c_t = w_t N_t + p^* \bar{x} + M_{t-1} \text{ and } p^* x_t + p_t c_t \leq M_{t-1},$$

where, the first one is the usual budget constraint and the last is the cash-in-advance constraint upon households' consumption purchases (i.e. since wages and profits are received only at the end of period t, a given period's expenditures must be financed out of money held at the beginning of the period). The solution to this problem is identical to the OG case that we consider as long as the cash-in-advance constraints are binding. This happens if along an equilibrium trajectory the following condition holds at all times:

$$\frac{dU}{dc_t} > \beta \frac{dU}{dc_{t+1}} \frac{p_t}{p_{t+1}}.$$

Since $\beta < 1$, the above condition is always satisfied at a steady-state solution. Therefore, for an equilibrium trajectory sufficiently close to the steady state, as we suppose in our local dynamic analysis, the condition holds as well.

From the definitions of $g(a)$ and $q(a)$, the elasticity of substitution $\sigma(a)$ between the traded and the nontraded goods can be defined as:

$$\frac{1}{\sigma(a)} \equiv \frac{\mathrm{d}\log q(a)/g(a))}{\mathrm{d}\log a} = \varepsilon_q(a) - \varepsilon_g(a) \tag{12}$$

where $\varepsilon_g(a) \equiv ag'(a)/g(a)$ and $\varepsilon_q(a) \equiv aq'(a)/q(a)$ are the elasticities of $g(a)$ and $q(a)$, respectively. Under Assumption 1 it follows that

Assumption 2: $u(a)$ is twice continuously differentiable, with $g(a) > 0$ and $g'(a) < 0$, for $a > 0$ and $\lim_{a\to 0} g(a)/q(a) = +\infty$, $\lim_{a\to +\infty} g(a)/q(a) = 0$.

Using this notation, the first-order conditions for the household problem are given by the budget constraints (10)–(11) and by the following expressions:

$$\frac{(1/B)V'(N_t/B)}{w_t} = \frac{q(a_{t+1})}{p^*}, \tag{13}$$

$$p_{t+1}c_{t+1} = \alpha(a_{t+1})M_t, \tag{14}$$

where $\alpha(a)$ represents the propensity to consume nontraded goods, which is defined through the following relationship:

$$g(a)/q(a) = \alpha(a)/a(1-\alpha(a)). \tag{15}$$

For later use, note that by use of Eqs. (11) and (14) we obtain that

$$\frac{p_{t+1}}{p^*} = \frac{\alpha(a_{t+1})}{a_{t+1}(1-\alpha(a_{t+1}))}. \tag{16}$$

Finally, note that a small α and a large σ tend to describe the 'potential' openness of the economy. From Eqs. (11) and (14) it can be seen that $(1-\alpha)$ is the propensity to consume traded goods, which is a commonly used proxy for the degree of openness. On the other hand, the elasticity of substitution in consumption, σ, measures the willingness of consumers to substitute nontraded with traded goods.

2.3. The equilibrium dynamic system

To obtain the equilibrium dynamic system of our model we proceed as follows. By use of Eqs. (8)–(10) and Eq. (14) we derive the first equation governing the dynamics:

$$m_t = \alpha(a_t)m_{t-1} + \bar{x} \quad \text{with } m \equiv M/p^*. \tag{17}$$

The above corresponds to the asset accumulation equation, reflecting the trade balance dynamics.[9] The second dynamic equation is obtained from Eqs. (8), (9), (13) and (14):

$$q(a_{t+1}) = \frac{\upsilon(N_t)}{\alpha(a_t)\, m_{t-1}} \quad \text{with } \upsilon(N) \equiv \frac{N}{B} V'(N/B). \tag{18}$$

To evaluate N_t, we put together Eqs. (8), (14) and (16):

$$N_t = [m_{t-1} a_t (1 - \alpha(a_t))]^{1/r}. \tag{19}$$

Thus, $N_t = N(m_{t-1}, a_t)$. Eqs. (17)–(19) govern the dynamics of the model: given the initial values for (m_{t-1}, a_t), we are able to determine (m_t, a_{t+1}). Indeed, by knowing (m_{t-1}, a_t) we obtain m_t through Eqs. (17) and (19) determines N_t. Then from Eq. (18) we obtain a_{t+1} since, by Assumption 1, $q(a)$ is an invertible function.

Definition: An intertemporal equilibrium with perfect foresight is a sequence $(m_{t-1}, a_t) \in R^2_{++}$, $t = 1, 2, \ldots, \infty$, such that

$$m_t = \alpha(a_t) m_{t-1} + \bar{x}, \quad q(a_{t+1}) = \frac{\upsilon[m_{t-1} a_t (1 - \alpha(a_t))]^{1/r}}{\alpha(a_t)\, m_{t-1}}. \tag{20}$$

Eq. (20) defines a two dimensional dynamic system, uniquely determined in the forward direction, with one predetermined variable[10] (m).

3. Steady-state analysis

In this section we will be studying the existence, unicity or multiplicity of stationary states for the dynamical system defined by Eq. (20). Our analysis will closely follow Cazzavillan et al. (1998) and we refer the reader to Section 3 of their paper for a thorough treatment of the issue.

[9] Note that in our simple model, there is no difference between the trade balance, the current account and the balance of payments. There is no international mobility of inputs, and money is the only asset varying only because of the trade balance.

[10] Note that at time t, m_{t-1} is determined by the state of the economy in the previous period, namely by the balance of trade in $t-1$. This implies that, although changes in expectations of the young at t (about $t+1$) will alter equilibrium at date t, the expenditure of the old is fixed and determined by past history.

3.1. Existence

First of all, note that a steady state ($m_{t-1} = m$, $a_t = a$) for all t, is a solution of Eq. (20) if and only if:

$$m(1 - \alpha(a)) = \bar{x}, \tag{21}$$

$$\upsilon((a\bar{x})^{1/r}) = m\alpha(a)q(a). \tag{22}$$

From Eq. (21) it follows that, as expected, the balance of trade is zero at a steady state (i.e. $x_t = \bar{x}$). Using Eqs. (8), (14), (16) and (21) we derive the steady-state level of employment, i.e.:

$$N^r = a\bar{x}. \tag{23}$$

In view of Eq. (15) and Eqs. (21)–(23) finding a steady state amounts to finding a value for $N > 0$ that satisfies:

$$\upsilon(N) = N^r g\left(\frac{N^r}{\bar{x}}\right) \quad \text{where } \upsilon(N) \equiv \frac{N}{B}V'(N/B). \tag{24}$$

Then, for a given $N > 0$ satisfying Eq. (24), m and a are uniquely determined by Eqs. (21) and (23).

In what follows we ensure the existence of a steady state, namely with $N = 1$, by choosing an appropriate value of the scaling parameter B, so that

$$\upsilon(1) = g\left(\frac{1}{\bar{x}}\right). \tag{24a}$$

From Assumption 1 on $V(\cdot)$ we know that the function $\upsilon(N) \equiv (N/B)[V'(N/B)]$ is increasing in N and decreasing in B; moreover, from Assumption 2, $g(1/\bar{x})$ is positive and defined over the interval $(0, +\infty)$. Hence, if $\lim_{N\to 0} \upsilon(N) < g(1/\bar{x}) < \lim_{N\to N^*} \upsilon(N)$, there is only one value for $B > 0$ such that Eq. (24a) is satisfied.

Proposition 1 (Existence of the steady state). *Under Assumption* 1 *on* $V(\cdot)$, *Assumption* 2 *on* $g(a)$ *and* $\lim_{N\to 0} \upsilon(N) < g(1/\bar{x}) < \lim_{N\to N^*} \upsilon(N)$, *let* B *be the unique solution of* $(1/B)V'(1/B) = g(1/\bar{x})$, *for a given value* $\bar{x} \in (0, +\infty)$; *then,* $(m, a) = (\bar{x}/(1 - \alpha(1/\bar{x})), 1/\bar{x})$ *is a stationary solution of the dynamical system* (20) *with* $N = 1$.

Before studying the issue of uniqueness and multiplicity of solutions to Eq. (24), it is useful to derive few relationships linking the elasticities of $g(a)$, $q(a)$, $\alpha(a)$ with $\sigma(a)$, and the expression defining the elasticity of the labour supply. From the obvious fact that $ag'(a) + q'(a) = 0$ and by use of Eq. (15), we obtain

$$\frac{\varepsilon_q(a)}{\varepsilon_g(a)} = -\left(\frac{\alpha(a)}{1-\alpha(a)}\right). \tag{25}$$

Hence, using Eqs. (12) and (25) we can easily derive an expression for the elasticity of $g(a)$, i.e.:

$$\varepsilon_g(a) = -\left(\frac{1-\alpha(a)}{\sigma(a)}\right). \tag{26}$$

From the definition of the propensity to consume nontraded goods, given in Eq. (15), and from Eq. (25) it follows that the elasticity $\varepsilon_\alpha(a) \equiv a\alpha'(a)/\alpha(a)$ is equal to $1-\alpha(a)+\varepsilon_g(a)$ which, by use of Eq. (26), corresponds to

$$\varepsilon_\alpha(a) = 1-\alpha(a))\left[1-\frac{1}{\sigma(a)}\right]. \tag{27}$$

From Eq. (13) we can easily derive the labour supply elasticity[11]which, from Assumption 1, is positive:

$$\varepsilon^s(N_t) = \frac{V'(N_t/B)B}{V''(N_t/B)N_t} > 0. \tag{28}$$

From Eq. (28), recalling the definition of $\upsilon(N)$ in Eq. (18), and in view of $\upsilon'(N) \equiv V'(N/B)(1/B) + (N/B)V''(N/B)(1/B)$, we can write

$$\frac{N\upsilon'(N)}{\upsilon(N)} = 1 + \frac{1}{\varepsilon^s(N)}. \tag{29}$$

3.2. Uniqueness versus multiplicity

So far we have shown that finding a stationary solution for the dynamics of the system requires finding a value $N > 0$ that satisfies Eq. (24) or, more simply, that satisfies

$$F(N) \equiv \frac{\upsilon(N)}{N^r g\left(\frac{N^r}{\bar{x}}\right)} = 1, \tag{30}$$

where, given our Assumptions 1 and 2, $F(N)$ is a continuous positively valued function for $N \in (0, N^*)$. Hence, studying the existence of multiple steady states and their numbers involves studying the number of solutions in N for Eq. (30). In particular, if $F(N)$ is a monotonic function (i.e. either $F'(N) > 0$ for

[11] Differentiating $(1/B)V' = q(a_{t+1})(p_t/p^*)(w_t/p_t)$. yields: $(1/B^2)V''\mathrm{d}N_t = q(a_{t+1}).(p_t/p^*)\mathrm{d}(w_t/p_t)$. This can be rewritten as $(1/B)V''\mathrm{d}N_t = V'(p_t/w_t)\mathrm{d}(w_t/p_t)$; therefore, $[(w_t/p_t)/N_t][\mathrm{d}N_t/d(w_t/p_t)] = BV'/V''N_t$.

$N \in (0, +\infty)$ or $F'(N) < 0$ for $N \in (0, +\infty)$) there exist *at most* one steady state, i.e. the one defined in Proposition 1. On the other hand, if $F'(N)$ changes its sign only once then, *at most* two steady-state solutions generically exist. Finally, if $F(N)$ is constant then, by Proposition 1, $F(N) = F(1) = 1$ for all values of N and there is a continuum of steady states (see Cazzavillan et al. 1998, Section 3).

To check whether $F(N)$ is monotonic or not we now analyse the sign of $F'(N)$. Differentiating $F(N)$ and by use of Eqs. (26) and (29), yields

$$F'(N) = \frac{Z(N)}{N} F(N)$$

where

$$Z(N) = \left[1 + \frac{1}{\varepsilon^s(N)} - r + r\left(\frac{1-\alpha(a)}{\sigma(a)}\right)\right] \text{for } a = \frac{N^r}{\bar{x}}. \tag{31}$$

Since $N > 0$ and $F(N) > 0$, studying the sign of $F'(N)$ is equivalent to studying the sign of the function $Z(N)$; the latter has the advantage of depending just on elasticities and other parameter values. The following propositions summarise the main results.

Proposition 2 (Uniqueness of the steady state). *Under the assumptions of Proposition* 1, *there is at most one steady state, with* $N = 1$, *if one of the following conditions is satisfied*:

(i) $(\sigma(a)/r)(1 + 1/\varepsilon^s(N)) + (1 - \sigma(a)) > \alpha(a) \Rightarrow Z(N) > 0$ *for all* $N > 0$,
(ii) $(\sigma(a)/r)(1 + 1/\varepsilon^s(N)) + (1 - \sigma(a)) < \alpha(a) \Rightarrow Z(N) < 0$ *for all* $N > 0$.

The proposition above shows that, when $\sigma(a)$, $\varepsilon^s(N)$ and $\alpha(a)$ are constant, either configuration (i) or (ii) is generically bound to occur and the steady state is unique. Note that the case of constant returns to scale falls into configuration (i). In fact, with $r = 1$, $Z(N) = 1/\varepsilon^s(N) + (1 - \alpha(a))/\sigma(a)$, which is always positive. Therefore, under constant returns only one steady state generically exists. However, as we shall see, with increasing returns to scale multiple steady-state equilibria are possible. In this case, a higher nontraded output enables economies of scale to be exploited, leading to a lower relative price and higher demand. Finally, notice that in the special case $Z(N) = 0$ for $N \in (0, N^*)$ a continuum of steady states will occur, if the assumptions of Proposition 1 are satisfied.

We address now the issue of multiplicity versus unicity by analysing in what circumstances the conditions stated in Proposition 2 are violated. We focus our attention on the case in which $F'(N)$ changes sign at most only once. In this case $F(N)$ is either single caved or single peaked, so that there are at most two steady states.[12] These are indeed the cases that arise in the constant elasticity specification that we are considering in Section 3.4.

[12] See Cazzavillan et al. (1998) for a thorough analysis of the issue.

Proposition 3 (Multiplicity of steady states). *Given Assumptions* 1 *and* 2, *there are at most two steady states if one of the following conditions is satisfied*:

(i) $[(1-\alpha(a))/\sigma(a)]$ *is increasing in a and* $\varepsilon^s(N)$ *is nonincreasing in N*,
(ii) $[(1-\alpha(a))/\sigma(a)]$ *is decreasing in a and* $\varepsilon^s(N)$ *is nondecreasing in N*.

Under case (i), $Z(N)$ is an increasing function. If $Z(0)<0$ and $Z(N^*)>0$ then $Z(N)$ changes its sign exactly once. Under case (ii), $Z(N)$ is a decreasing function. Hence, if $Z(0)>0$ and $Z(N^*)<0$, $Z(N)$ also changes its sign exactly once.

Therefore, if the assumptions of Proposition 1 are satisfied and $Z(N)$ do not vanish at $N=1$ (i.e. $Z(1)\neq 0$), there are exactly two steady states whenever either (i) or (ii) of Proposition 3 holds, provided that appropriate boundary conditions are satisfied (namely that $F(N)-1$ has the same sign for N close to zero and N close to N^*).

In the case where $Z(N)$ vanishes for $N=1$ only one steady state exists. However, as discussed in Cazzavillan et al. (1998) the uniqueness of the steady state does not persist. Therefore we can treat it as a nongeneric case. Later on in the paper, where the local dynamics and bifurcations around the steady state are studied (see Section 4 below), it can be checked that the case just described corresponds to the occurrence of a transcritical bifurcation (i.e. exchange of stability properties between two steady states). In other words, by slightly changing the value of $\varepsilon^s(1)$, $Z(1)$ crosses the value zero and two steady state coexist.

3.3. Welfare analysis of the multiple steady states

In this section we analyse the welfare properties of the two steady states studied in the section above, and check if one Pareto dominates the other. Assuming that Proposition 3 holds, and letting (m_1, a_1) and (m_2, a_2) be two steady states, it can be shown that the steady state with a lower a is Pareto-dominated by the other. Suppose that $a_1>a_2$. From the equilibrium conditions (23) and (8), and recalling that at a steady state $x_t=\bar{x}$, it follows that $N_1>N_2$ and $c_1>c_2$. Hence, consumption of the old households is higher in the first steady state and so it is their welfare. Moreover, the offer curve (in N and c) of the young generation for steady-state solutions ($x_t=\bar{x}$) is defined by $\upsilon(N)=ag(a)\bar{x}$, where $a=c/\bar{x}$. Since $\upsilon(N)$ is positively sloped, the offer curve is also positively sloped if $(1-\alpha(a))/\sigma(a)<1$, for all $a>0$ (i.e. from Eq. (26): $1+\varepsilon_g(a)>0$). Therefore, in this case, welfare of the young is also higher in the first steady state.

Proposition 4. *Under the assumptions of Proposition* 3, *let* (m_1, a_1) *and* (m_2, a_2) *be two steady states. Given Assumptions* 1 *and* 2 *and assuming further that* $(1-\alpha(a))/\sigma(a)<1$, *for* $a>0$, *it follows that* (m_1, a_1) *Pareto-dominates* (m_2, a_2) *for* $a_1>a_2$.

3.4. The case of CES economies

We now analyse the case where preferences display constant elasticities. In particular, we assume the following utility functions:

$$u(a) = [sa^{(\sigma-1)/\sigma} + (1-s)]^{\sigma/(\sigma-1)} \quad \text{where} \quad 0 < s < 1, \sigma > 0 \tag{32}$$

and

$$V\left(\frac{N}{B}\right) = \frac{1}{b}\left(\frac{N}{B}\right)^b \quad \text{where} \quad b = 1 + \frac{1}{\varepsilon^s}, \quad B > 0. \tag{33}$$

Where, Assumptions 1 and 2 are all satisfied; and the scaling parameter B is fixed so that $(m, a) = (\bar{x}/(1-\alpha(1/\bar{x})), 1/\bar{x})$ is a steady state (as from Proposition 1). In this case, the propensity to consume nontraded goods is given by

$$\alpha(a) = \frac{s}{s + (1-s)a^{(1-\sigma)/\sigma}} \tag{34}$$

which is increasing (decreasing) in a for $\sigma > 1$ ($\sigma < 1$). One can also easily check that the function $Z(N)$ defined in Eq. (31) is increasing, constant or decreasing in a depending on whether $\sigma < 1$, $\sigma = 1$, $\sigma > 1$. We analyse these three cases in turn:

$\sigma < 1$: $\alpha(a)$ decreases from 1 to 0 and $Z(N)$ goes from $(b-r)$ to $b - r(\sigma-1)/\sigma$ as a increases from 0 to $+\infty$. Therefore, if $b > r$ case (i) of Proposition 2 applies and the steady state is unique. For $b < r$, $Z(N)$, being an increasing function (case (i) of Proposition 3 applies), changes its sign from negative to positive exactly once. Therefore, provided that $Z(N)$ does not vanish at $N = 1$, there are exactly two steady states, since boundary conditions are satisfied.[13]

[13] Checking if the appropriate boundary conditions hold implies checking if $\lim_{N\to 0} F(N) > 1$ and $\lim_{N\to+\infty} F(N) > 1$ are satisfied when $F(N)$ is single caved, and if $\lim_{N\to 0} F(N) < 1$ and $\lim_{N\to+\infty} F(N) < 1$ hold when $F(N)$ is single peaked. With CES preferences the function $F(N)$ becomes (see Eq. (30))

$$F(N) = \frac{N^{b-r}}{\left[s + (1-s)\left(\dfrac{N^r}{\bar{x}}\right)^{(1-\sigma)/\sigma}\right]^{-1/(1-\sigma)} sB^b}.$$

With $\sigma < 1$ and $b < r$, the function $F(N)$ is single caved (case (i) Proposition 3 holds) and, it can be easily checked that $\lim_{N\to 0} F(N) = \lim_{N\to+\infty} F(N) = +\infty$. Therefore, the single caved function $F(N)$ has the same sign when is close to the lower and upper bounds of N and there are exactly two steady states (ignoring the nongeneric case of $F(N) = 0$ when $N = 1$). See Cazzavillan et al. (1998) for further details.

$\sigma > 1$: $\alpha(a)$ increases from 0 to 1 and $Z(N)$ goes from $b - r(\sigma - 1)/\sigma$ to $(b - r)$ as a moves from 0 to $+\infty$. As before, if $b > r$ case (i) of Proposition 2 applies and the steady state is unique. If $b < r(\sigma - 1)/\sigma$, again there is only one steady state (case (ii) of Proposition 2 applies). Whereas, if $r(\sigma - 1)/\sigma < b < r$ two steady states generically exist (provided that $Z(1) \neq 0$) since the appropriate boundary conditions are satisfied.[14] Note that, in this case, $Z(N)$ is a decreasing function (case (ii) of Proposition 3) changing its sign from positive to negative values exactly once.

$\sigma = 1$: $\alpha(a) = s$. Since $Z(N)$ is constant and Proposition 2 applies. There is a unique steady state (or a continuum of steady states in the nongeneric case $Z(N) = 0$).

For $\sigma \neq 1$ multiple equilibria arise when the elasticity of labour supply is larger than the elasticity of labour demand (i.e. the labour supply schedule is flatter than the labour demand schedule and the latter is upward sloping due to increasing returns). Consider the case $\sigma < 1$: under that configuration the condition $b < r$ corresponds to $1/\varepsilon^{s} < (r - 1) = 1/\varepsilon^{d}$. Therefore, so long as the economy exhibits some degree of increasing returns to scale the stationary state may not be unique so that indeterminacy emerges.[15]

4. Local dynamics and bifurcation analysis

In this section we analyse the role of openness[16] (and increasing returns to scale) on the emergence of local endogenous fluctuations. Assuming that a steady state exists (Proposition 1), we proceed to local stability and bifurcation analysis by use of the geometrical method developed by Grandmont et al. (1998). The latter applies to discrete time nonlinear two-dimensional economic systems; does not require any particular preferences and technology specification and provides a full characterization of the equilibrium trajectories around a steady state. In particular, we will show that

(1) When the economy is sufficiently open, local deterministic endogenous fluctuations (cycles of period two) emerge into the system.

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[14] In that case, with $\sigma > 1$ and $r(\sigma - 1)/\sigma < b < r$, the function $F(N)$ is single peaked (case (ii) Proposition 3 holds) and $\lim_{N\to 0} F(N) = \lim_{N\to +\infty} F(N) = 0$. Thus, the required boundary conditions are satisfied (see footnote 13).

[15] On the issue of indeterminacy and increasing returns (both internal and external to the firm) see, in particular, Benhabib and Farmer (1994) and Farmer and Guo (1994).

[16] Recall that high values of σ and low values of α are here considered as symptoms of highly open economies.

(2) The emergence of local stochastic endogenous fluctuations is influenced by the degree of openness in a mixed way: while high values of σ help the occurrence of local indeterminacy, low values of α make its occurrence more difficult.
(3) Increasing returns to scale are necessary for the emergence of local (deterministic and stochastic) endogenous fluctuations, as it would be in the equivalent closed economy set-up.

This section will be divided into three parts: first, we will outline the general method; second, we will apply the method to our model; thirdly, we will explore how openness affects the equilibrium dynamics of the economy.

4.1. The general framework

The implementation of the method that we are going to use involves four steps: (i) expressing the determinant (D) and trace (T) of the Jacobian matrix, for the dynamic system under analysis, as a function of the relevant 'parameters' of the model; (ii) obtaining the relationship, say Δ, linking T and D for different given values of one of the parameters, say λ, i.e. the points $(T(\lambda), D(\lambda))$ where λ is the chosen bifurcation parameter; (iii) representing the locus of points $(T(\lambda), D(\lambda))$ in the space (T, D) and studying its behaviour as λ changes; (iv) analysing how the relationship Δ is moving in the space (T, D) with changes in the values of the other relevant parameters of the model.

In what follows we briefly explain the importance of steps (iii) and (iv), which are indeed the core of the geometrical method. A detailed application of the methodology to our particular model is given in Sections 4.2 and 4.3.

To study local stability, we follow the usual procedure of analysing the eigenvalues of the Jacobian matrix (evaluated at a steady-state solution). In particular, note that for two-dimensional dynamic systems we can always define three relevant lines in the space (T, D): line AC ($D = T - 1$), line BC ($D = 1$, $|T| < 2$) and line AB ($D = -(T + 1)$). See for instance Fig. 1 (disregarding the line Δ_1, for the moment) in Section 4.2.

The lines AB, BC and AC in Fig. 1 divide the (T, D) space into three different regions according to the dynamic properties of the steady state (e.g. source, sink, saddle),[17] and enable us to study the emergence of local endogenous fluctuations (stochastic and deterministic).

As regard to stochastic endogenous fluctuations we know that, if a steady state is a sink, local stochastic equilibria driven by self-fulfilling expectations can arise. In our model this corresponds to indeterminacy of the steady state, given

[17] See Azariadis (1993), (pp. 63,64), for details.

that we have only one predetermined variable (money).[18] The three lines defined above are also important to identify the occurrence of bifurcations through which deterministic endogenous fluctuations may emerge. A *bifurcation* occurs when, by continuously changing the value of one parameter[19] of a nonlinear dynamic model (the bifurcation parameter), there is a qualitative change in the dynamic properties of a steady state. Therefore, if by slightly changing the bifurcation parameter, a pair of conjugate complex eigenvalues crosses the unit circle, the values of T and D cross the BC line in its interior. Then a steady state which was a sink becomes a source or vice versa. In this case a *Hopf* bifurcation occurs (deterministic equilibrium trajectory lying on an invariant closed curve nearby a steady state). On the other hand, if the values of T and D cross the AB line, an eigenvalue is crossing the value -1, and a *flip* bifurcation occurs (deterministic cycle of period two nearby a steady state). Finally, if the values of T and D cross the AC line, an eigenvalue is crossing the value 1, and a *transcritical*[20] bifurcation occurs (two steady states exchanging stability properties).[21] Furthermore, as shown in Grandmont et al. (1998), we are also able to study the emergence of stochastic equilibria nearby local bifurcations. In fact, if the Hopf (flip) bifurcation is supercritical[22] then, there are infinitely many stochastic equilibria containing in their interior the invariant closed curve (the cycle of period two). Similarly, in the case of a transcritical bifurcation if the steady state considered in our local analysis is the saddle one (determinate) while the other is a sink (indeterminate), then there are no stochastic fluctuations arbitrarily near it, but there are infinitely many stochastic equilibria nearby the other steady state.[23]

[18] See Woodford (1986) and Guesnerie and Woodford (1992) for more details. In short, indeterminacy will arise when the number of stable eigenvalues (i.e. roots strictly lower than one in absolute value) is larger than the number of predetermined variables. In our model this means that, for a given initial condition m_{t-1} close to the steady state, there exists an infinite number of asset accumulation (trade balance) paths converging to the same balanced trade steady state.

[19] Notice that we exclusively consider *codimension one* bifurcations by varying only one parameter.

[20] In general, when one root is crossing the value 1 different types of local bifurcations (transcritical; generic saddle-node or pitchfork) occur, depending on the properties and number of stationary states (see Grandmont (1988), (Section C)). Since we have assumed that the steady state persists (Proposition 1 holds) for all parameter values under consideration, saddle-node bifurcations are ruled out. Moreover, from Proposition 2 it follows that there exists at most two steady states which implies that pitchfork bifurcations cannot occur either.

[21] Indeed, other conditions must also be satisfied for the occurrence of bifurcations (see, for instance, the bifurcation theorems stated in Grandmont (1988) or Hale and Koçak (1991)). However, these other conditions are generically satisfied.

[22] A Hopf (flip) bifurcation is supercritical if the invariant closed curve (cycle of period two) is stable, and therefore indeterminate, while the steady state nearby is determinate.

[23] Notice that, if we had not taken into account the nonlinearities of the model, we would had not been able to detect the existence of endogenous fluctuations (deterministic and stochastic) whenever the steady state under analysis is locally determinate.

Since the trace and determinant depend also on other parameters besides the one chosen as bifurcation parameter, the local dynamics of the model is also determined by the values taken by the other parameters. That explains why we should also study how the relationship Δ moves when we change the other parameters of the model (step (iv)).

4.2. Implementation

Following the four steps highlighted in the previous section, we begin by deriving the Jacobian matrix, evaluated at a steady-state solution[24] for the dynamic system (20). Using the relationships defined in Eqs. (25)–(29), the matrix J is given by

$$J = \begin{bmatrix} \alpha & \alpha\left(1 - \dfrac{1}{\sigma}\right) \\ \dfrac{\sigma}{\alpha}\left(\dfrac{1}{r}\left(1 + \dfrac{1}{\varepsilon^{s}}\right) - 1\right) & \dfrac{1}{\alpha}\left(\dfrac{1}{r}\left(1 + \dfrac{1}{\varepsilon^{s}}\right)(\sigma(1-\alpha) + \alpha) - (1-\alpha)(\sigma - 1)\right) \end{bmatrix} \tag{35}$$

where $r \geq 1$ is the degree of returns to scale; $0 < \alpha < 1$ is the propensity to consume nontraded goods; $\sigma > 0$ is the elasticity of substitution in consumption between traded and nontraded goods and $\varepsilon^{s} > 0$ is the elasticity of labour supply. The expressions for the determinant and trace of the Jacobian matrix (35) can be written as follows:

$$D = \frac{1}{r}\left(\frac{1}{\varepsilon^{s}}\right) + D_1 \quad \text{where } 0 < D_1 = \frac{1}{r} < 1, \tag{36}$$

$$T = \frac{1}{\varepsilon^{s}}\left[\frac{\sigma(1-\alpha) + \alpha}{\alpha r}\right] + T_1 \tag{37}$$

where $T_1 = \alpha + \dfrac{1}{\alpha r}[1 + (r-1)(1-\alpha)(1-\sigma)]$.

Consider fixed values for α, σ and r, and choose $\varepsilon^{s} \in (0, +\infty)$ as the bifurcation parameter. Then, from Eqs. (36) and (37), the locus $(T(\varepsilon^{s}), D(\varepsilon^{s}))$ is defined through the following relationship:

$$D = \Delta(T) = (T - T_1)\left[\frac{\alpha}{\sigma(1-\alpha) + \alpha}\right] + D_1 \quad \text{for } T > T_1.$$

[24] Hence α, σ and ε^{s} are all evaluated at the steady state defined in Proposition 1.

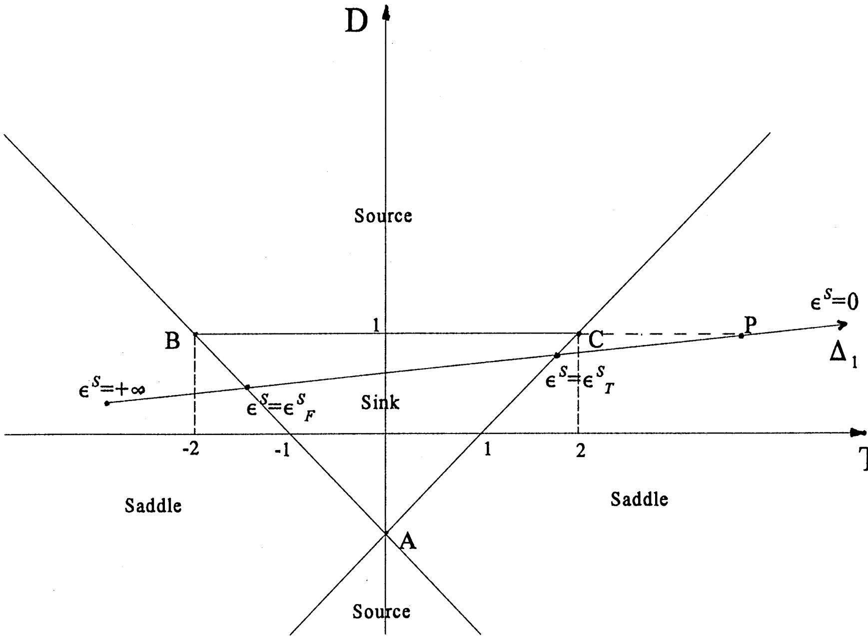

Fig. 1. The half-line Δ in the (T, D) plane.

The expression Δ is linear and is always positively sloped, i.e.:

$$\Delta' = \frac{\partial D/\partial \varepsilon^s}{\partial T/\partial \varepsilon^s} = \frac{\alpha}{\sigma(1-\alpha)+\alpha} > 0 \quad \text{where } 0 < \Delta' < 1. \tag{38}$$

In graphical terms, the locus $(T(\varepsilon^s), D(\varepsilon^s))$ with $\varepsilon^s \in (0, +\infty)$ is a half-line Δ beginning, for $\varepsilon^s = +\infty$, in (T_1, D_1) and pointing upwards. An example is given in Fig. 1.

Before proceeding further, note that in our model Hopf bifurcations are always ruled out. The half-line Δ always crosses the point $P = (\alpha + 1/\alpha, 1)$ and, since $\alpha + 1/\alpha > 2$, it cannot cross BC in its interior (see Fig. 1). However, flip and transcritical bifurcations may occur, namely when ε^s crosses either the value ε^s_F or the value ε^s_T.[25]

Having established the relationship Δ linking $T(\varepsilon^s)$ and $D(\varepsilon^s)$, we can now proceed to studying how local dynamics and bifurcations are affected by movements in the other parameter values of the model. In the following

[25] $\varepsilon^s_F(\varepsilon^s_T)$ is the value of the parameter ε^s such that the half-line Δ crosses the AB (AC) line. The corresponding analytical expressions are specified at the end of Proposition 5.

subsection we focus in particular on the role played by the degree of openness (measured by α and σ).

4.3. The role of openness

To analyse how the half-line Δ moves in the space (T, D) as α changes in (0,1), we simply need to understand how Δ' and (T_1, D_1) are affected by α.

From Eq. (38) the slope of the half-line Δ is always increasing in α, moving from 0 to 1 as α increases from 0 to 1. As regard to the initial point (T_1, D_1), this always lies – given a fixed value of $r > 1$ – on the same horizontal line (between BC and the axis T, e.g. the dotted line in Fig. 2), since $D_1 = 1/r$ does not depend on α. T_1, on the other hand, depends on α, $T_1(\alpha)$; decreasing for $\alpha < \alpha_a = [(r - \sigma(r-1))/r]^{1/2}$ and increasing for $\alpha > \alpha_a$. Note that, in the limiting case of a closed economy ($\alpha = 1$), the half-line Δ lies on the AC line. In fact, Δ' $(\alpha = 1) = 1$ and T_1 $(\alpha = 1) = 1 + D_1$.

Note further that $\alpha_a \leq 0$ when $\sigma \geq r/(r-1)$, in which case T_1 is always increasing in $\alpha > 0$. See top diagram in Fig. 2. Here, the initial point (T_1, D_1) moves on the horizontal line $D = D_1$, with T_1 going from $-\infty$ (for α close to 0, hence Δ' close to 0) until it reaches the AC line (for α close to 1, hence Δ' close to 1), crossing the AB line for $\alpha = \alpha_F$.[26] Take for instance the half-line Δ_1, the case of $\alpha < \alpha_{\mathrm{F}}$ (which is also reproduced in Fig. 1). Here, the steady state is a saddle for $\varepsilon^{\mathrm{s}}_{\mathrm{F}} < \varepsilon^{\mathrm{s}} < +\infty$. A flip bifurcation occurs when ε^{s} crosses the value $\varepsilon^{\mathrm{s}}_{\mathrm{F}}$ and the steady state becomes a sink for $\varepsilon^{\mathrm{s}}_{\mathrm{T}} < \varepsilon^{\mathrm{s}} < \varepsilon^{\mathrm{s}}_{\mathrm{F}}$. Then as ε^{s} crosses the value $\varepsilon^{\mathrm{s}}_{T}$ a transcritical bifurcation occurs and the steady state becomes again a saddle for $\varepsilon^{\mathrm{s}} < \varepsilon^{\mathrm{s}}_{\mathrm{T}}$.

We turn now to the case of $\sigma < r/(r-1)$, depicted in the bottom diagram of Fig. 2. In this case T_1 is always positive but nonmonotonic in α. For $0 < \alpha < \alpha_a$, the initial point moves to the left on the horizontal line, $D = D_1$. Indeed T_1 goes from $+\infty$ (for α close to 0, hence Δ' close to 0), crosses the AC line for $\alpha = \alpha_T$ until it reaches its minimum value for $\alpha = \alpha_a$ (where $0 < \Delta' < 1$). Then, as α increases from α_a to 1, the half-line Δ becomes steeper while T_1 moves to the right until it hits the line AC. By direct inspection of the bottom diagram in Fig. 2, it is clear that flip bifurcations are ruled out, while transcritical bifurcations only occur for $\alpha > \alpha_T$. Finally, for $\alpha < \alpha_T$ the steady state is always a saddle.

The following proposition summarises the results on local stability and bifurcation analysis.

Proposition 5. *Let r be the degree of increasing returns*; *α, σ and ε^{s} be, respectively, the propensity to consume nontraded goods, the elasticity of substitution in*

[26] The expression for α_{F} and for α_{T}, defined below in the text, is given at the end of Proposition 5.

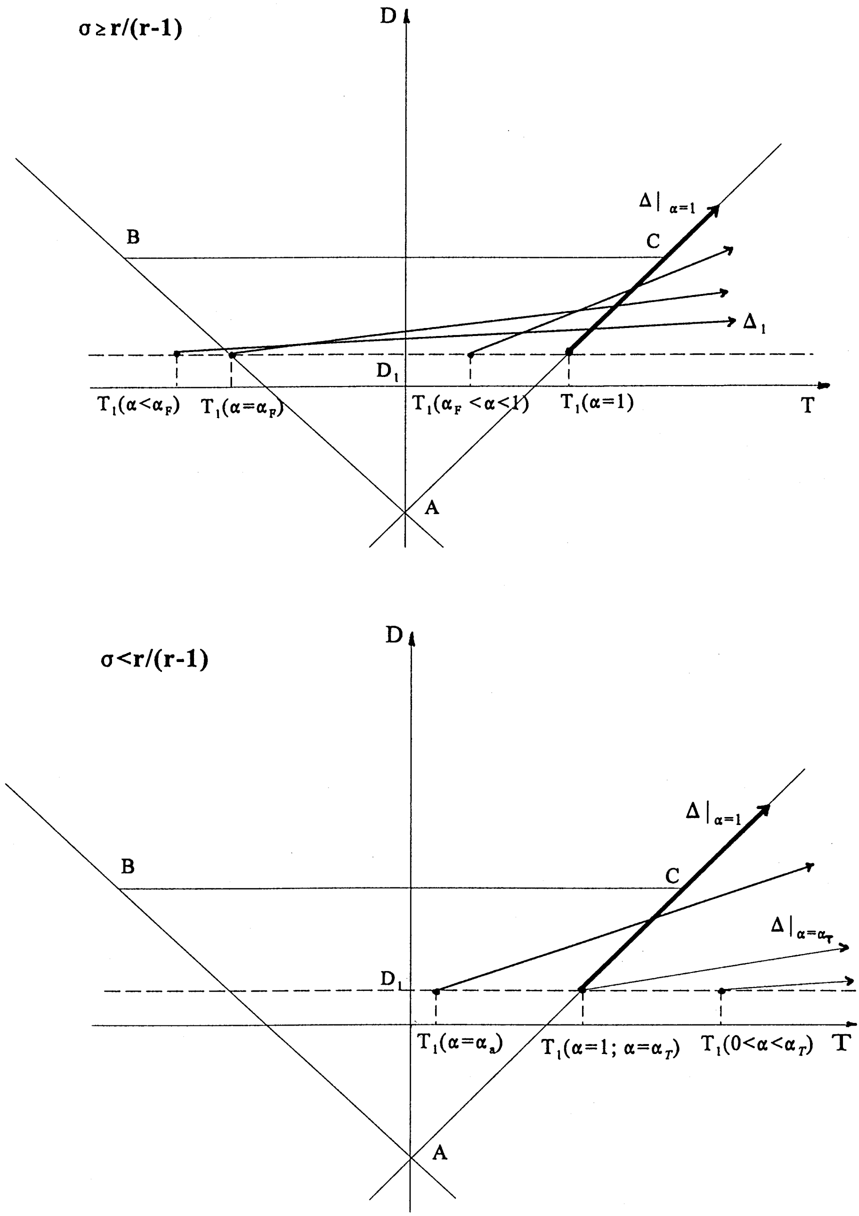

Fig. 2. Shifts in half-line Δ with changes in $\alpha \in (0, 1)$.

consumption between traded and nontraded goods and the elasticity of labour supply, all evaluated at the steady state referred in Proposition 1. *Then the following generically holds.*

(I) *if* $\sigma \geq r/(r-1)$ *then*:
  (i) *for* $\alpha < \alpha_F$, *the steady state is a saddle when* $0 < \varepsilon^s < \varepsilon^s_T$; *undergoes a transcritical bifurcation for* $\varepsilon^s = \varepsilon^s_T$ *becoming a sink for* $\varepsilon^s_T < \varepsilon^s < \varepsilon^s_F$; *undergoes a flip bifurcation for* $\varepsilon^s = \varepsilon^s_F$ *becoming a saddle for* $\varepsilon^s > \varepsilon^s_F$.
  (ii) *for* $\alpha > \alpha_F$, *the steady state is a saddle when* $0 < \varepsilon^s < \varepsilon^s_T$; *undergoes a transcritical bifurcation for* $\varepsilon^s = \varepsilon^s_T$ *becoming a sink for* $\varepsilon^s > \varepsilon^s_T$.

(II) *if* $\sigma < r(r-1)$ *then*
  (i) *for* $\alpha < \alpha_T$, *steady state is always a saddle.*
  (ii) *for* $\alpha > \alpha_T$, *the steady state is a saddle when* $0 < \varepsilon^s < \varepsilon^s_T$; *undergoes a transcritical bifurcation for* $\varepsilon^s = \varepsilon^s_T$ *becoming a sink for* $\varepsilon^s > \varepsilon^s_T$.

where[27] $\varepsilon^s_T = 1/(b_1 - 1)$; $\varepsilon^s_F = 1/(b_2 - 1)$, *with* $b_1 \equiv r[(\sigma - (1-\alpha))/\sigma]$ *and* $b_2 \equiv r[1 - (1+\alpha)/(\sigma(1-\alpha) + 2\alpha]$, $\alpha_T = (r - (r-1)\sigma/r$; $\alpha_F \{ -A + [A^2 - 4r(1-(r-1)(\sigma-1))]^{1/2}\}/2r$, *with* $A \equiv (1+r) + (r-1)(\sigma-1)$.

Proposition 5 states that endogenous fluctuations (deterministic and stochastic) emerge within our model. In the case in which a flip bifurcation occurs local deterministic fluctuations emerge, namely cycles of period two nearby the steady state. In the simulations we have made this bifurcation was supercritical, i.e. a cycle of period two appeared when the steady state was a saddle (for values of ε^s close but higher than ε^s_F). As explained before, this means that the cycle is indeterminate and there are lots of stochastic endogenous fluctuations around it. Moreover, when the steady state is a sink (indeterminate) there are also infinitely many stochastic endogenous fluctuations arbitrarily near the steady state. In the case of a transcritical bifurcation in which two steady states coexists (when ε^s is close enough to ε^s_T), one being a saddle and the other a sink (indeterminate), there are again lots of stochastic endogenous fluctuations arbitrarily near the sink. Hence, even when the steady state under consideration is a saddle (determinate), endogenous fluctuations emerge if ε^s is close enough to the 'sink boundaries' (ε^s_T and ε^s_F).

The above findings on local stability, bifurcations and on the emergence of endogenous fluctuations are depicted in Fig. 3, which plots ε^s_T and ε^s_F as functions of α. This picture represents: (i) the locus of points (α, ε^s) such that a steady

[27] It is worth noticing that when ε^s takes the value ε^T_s the function $Z(N)$ (see Eq. (31)), evaluated at the steady state (defined in Proposition 1), takes the value zero (i.e. $Z(1) = 0$). This means that when $\varepsilon^s = \varepsilon^s_T$ only one steady state exists, but for ε^s close enough to ε^s_T two steady states coexist.

state is a sink (shaded areas) or a saddle; (ii) for each given value of α, the values of ε^s such that bifurcations occur (lines in bold).

It is instructive to analyse first the limiting case of closed economy ($\alpha = 1$). Recall that under this assumption the half-line Δ lies on the AC line and no flip bifurcation can occur.[28] However, as can be seen in Fig. 3, the steady state is indeterminate for $\varepsilon^s > 1/(r-1) = \varepsilon^s_T(\alpha = 1)$ and local stochastic endogenous fluctuations can arise; whereas, a transcritical bifurcation occurs when ε^s crosses the value $\varepsilon^s_T(\alpha = 1)$.

From the top diagram in Fig. 3 we can see that a minimum degree of openness, namely $\alpha < \alpha_F$ (and $\sigma \geq r/(r-1)$), is required for flip bifurcations to occur. Moreover, a high degree of openness (i.e. low α) facilitates the occurrence of flip bifurcations since the value of ε^s needed for the occurrence of a flip bifurcation is lower.[29] On the contrary, from both diagrams in Fig. 3, it can be seen that high values of α help indeterminacy of the steady state and the emergence of local stochastic endogenous fluctuations. Indeed, the range of values for ε^s such that indeterminacy can arise widens up as α increases.

As for the role of σ, in all cases analysed a high elasticity of substitution in consumption between traded and nontraded goods helps the emergence of endogenous fluctuations. In fact: (i) ε^s_T and ε^s_F are decreasing in σ; (ii) α_T decreases as σ increases; (iii) flip bifurcations only occur for sufficiently high value of σ.

Finally, we notice that increasing returns are necessary for the emergence of local (deterministic or stochastic) endogenous fluctuations. If $r = 1$ the horizontal asymptote $(1/r - 1)$ will go to infinite and the steady state will always be a saddle. However, it can be shown[30] that high openness requires low degree of increasing returns for flip bifurcations to occur at a given fixed value of ε^s.

5. Discussion

The previous section has considered the mathematical analysis of the dynamics of the economic system. In this section, we aim to discuss its economic

[28] See Lloyd-Braga (1995a) where the closed economy case is considered. There, it is shown that even under increasing returns to scale flip bifurcations cannot occur for $\varepsilon^s > 0$. Indeed, in the limit case of a closed economy the equilibrium system degenerate into a one-dimensional dynamic system with no predetermined variables (no trade balance in a closed economy): the eigenvalue crossing the value 1 for ε^s crossing $\varepsilon^s_T = 1/(r-1)$.

[29] Indeed empirical studies seem to reject a high value for the elasticity of labour supply.

[30] From Eq. (39) we can see that, when $\sigma \geq r/(r-1)$, ε^s_F increases as r decreases whereas it decreases as α decreases (high openness).

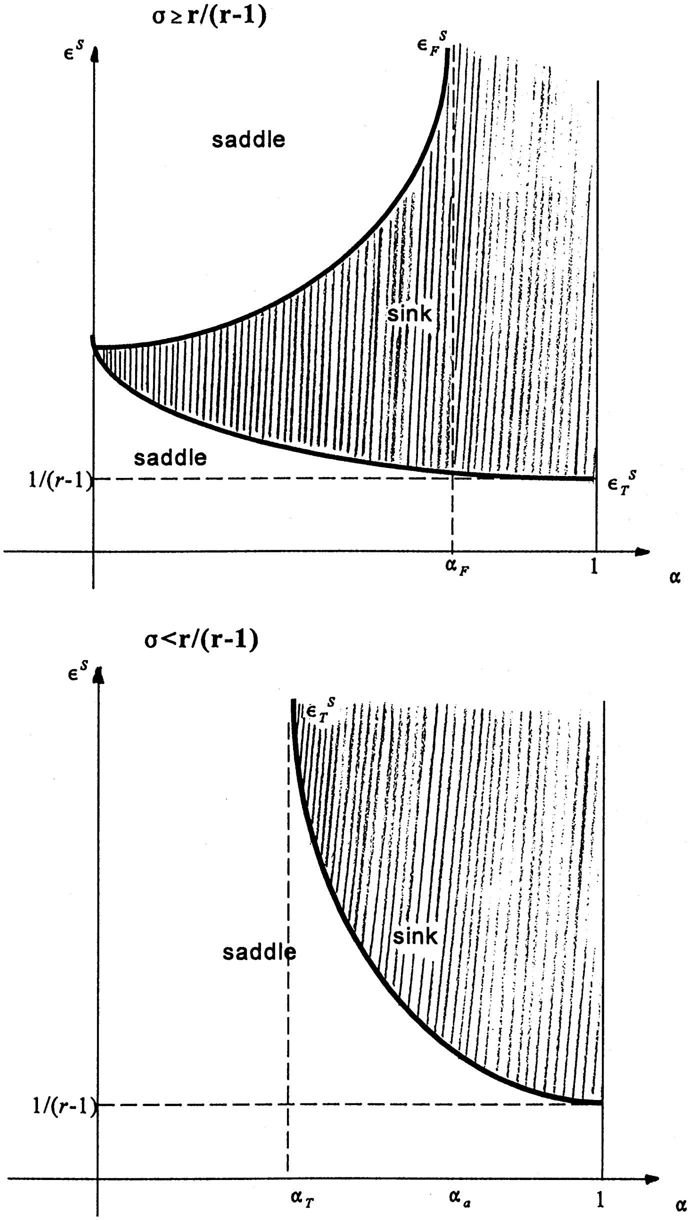

Fig. 3. Local dynamics and bifurcations in the space (α, ε^s).

significance. As shown in Section 4.3 the existence of increasing returns is necessary for the emergence of local endogenous fluctuations, either deterministic or stochastic; whereas the occurrence of flip bifurcations and the emergence of deterministic cycles is only possible for an open economy.

To give a clear intuition of these results it is useful to analyse first the benchmark case of a closed economy with constant returns. In this case a positively sloped labour supply curve (current leisure and future consumption are gross substitute) implies an upward sloping offer curve in the space of current work and future consumption. Suppose that the economy is at its steady-state solution for sometime and that, for some reason, the expected future price increases. The labour supply curve will then shift to the left (due to the gross substitution assumption) and, since real wages are constant under CRS, employment will decrease. Under perfect foresight, the increase in future prices will also decrease future consumption[31] and therefore will also decrease employment in the next period (needed for its production). To get local endogenous fluctuations the equilibrium dynamic system should be able to produce nonmonotonic trajectories, which is ruled out with constant returns.

If there are increasing returns and at the steady state the labour supply is more elastic than the labour demand curve ($\varepsilon^{s} > \varepsilon^{d} = 1/r - 1$), then the expected future increase in prices will call for an increase in employment (together with an increase in real wages). Meanwhile, due to the increase in future prices, future consumption decreases together with employment in the next period. Thus this economy seems to be able to produce equilibrium trajectories for labour that are nonmonotonic (i.e. local endogenous fluctuations nearby the steady state). However, the latter cannot be deterministic (perfect foresight) equilibrium trajectories. The initial increase in employment is not compatible with the decision of lowering future consumption for a rational consumer acting under perfect foresight, since the offer curve is positively sloped. Nevertheless, as shown in our analysis, local endogenous fluctuations can be of stochastic nature.

Turning to the open economy case, we show that the system can produce *deterministic* endogenous fluctuations (cycles of period two) as well as stochastic endogenous fluctuations. In an open economy, in fact, a simultaneous decrease in future consumption (of the nontraded good) and an increase in current employment (and real wage income), following an increase in the expected future price of the nontraded good (p_{t+1}), becomes compatible with the perfect foresight hypothesis. This is obtained in our model because the offer curve, in the space (N_t, c_{t+1}), is parametrized in the consumption of the traded good (x_{t+1}), and will shift to the right when x_{t+1} increases. In our economy the increase in x_{t+1}, following the increase in p_{t+1}, is higher the higher is the substitution effect

[31] Recall that in a closed economy $c_{t+1} = M/p_{t+1}$.

(i.e. high σ) and the higher is the income effect (i.e. low α). That explains why the more open is the economy the higher the possibilities for the occurrence of flip bifurcations and for the emergence of local deterministic fluctuations.

It is important to emphasise that whenever the system displays nonmonotonic equilibrium trajectories of the type described above, the balance of trade of our economy is persistently fluctuating (switching from a position of net exporter to net importer and vice versa).

Lastly, what is the empirical plausibility of our model? Clearly, it is difficult to relate such a stylised model to the real economy. Our aim has not been to develop a model which can then be directly calibrated and used to understand an actual economy: we have left many important features out of the model that are clearly important in the real world. The value of α can take a wide range of values in different economies, depending on their degree of openness. The estimates of σ vary from 0.44 (Tesar, 1993) to as high as 1.28 for some countries (Mendoza, 1995). There are a variety of estimates for r, which we might expect to be above 1 but not by much (see e.g. Caballero and Lyons, 1990). However, our model does show that the combination of increasing returns and the openness of the economy makes deterministic endogenous fluctuations possible and, in general, makes the taste and preference parameters' requirement for such fluctuations to arise less restrictive. If we introduced other realistic features such as capital accumulation, then the scope for endogenous fluctuations would be increased: in general, the intertemporal relationships would be more complicated, allowing for more nonmonotonicity and a wider range of parameters to vary.

6. Conclusion

In this paper we develop a standard overlapping generation model of a small open economy with a traded and a nontraded good, where labour is the only factor of production and money the only asset. The nontraded good sector is characterised by Cournotian-monopolistic competition with free entry and increasing returns to scale at the firm level, whereas output in the traded good sector is exogenous. In this setting the dynamic equilibrium is described by a two-dimensional system with one predetermined variable (money), which we analyse for the emergence of local deterministic and stochastic endogenous fluctuations.

The paper demonstrates the possibility of deterministic (and stochastic) endogenous fluctuations in an open economy with increasing returns to scale. Under constant returns and positive labour supply elasticity, the system is always at a saddle point stable equilibrium. Hence, as in closed economies (see e.g. Lloyd-Braga (1995a), Cazzavillan et al. (1998)), increasing returns (either internal or external) is one crucial element leading to the emergence of endogenous fluctuations.[32] The new feature of the paper has been to introduce openness into the economy: contrary to the equivalent closed economy set up, households can substitute consumption between traded and nontraded goods, and deterministic cycles can emerge.

Clearly, the simplicity of the underlying economic model used does not permit us to make any specific observations about the functioning of real economies. However, we have demonstrated that there is a potential solution to the problem of the excess variability in the current account to which other approaches have given rise. An economy in which all of the fundamentals are unchanged can still exhibit persistent volatility in its current account. This is perhaps the clearest conclusion to be drawn from our paper.

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[32] Allowing for imperfect competition in both output and labour markets may reduce the role of increasing returns. As shown by Jacobsen (1997), endogenous fluctuations still arise with a positively sloped labour supply schedule and constant returns to labour technology.

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