

INTEGER PRICING AND BERTRAND-EDGEWORTH OLIGOPOLY WITH STRICTLY CONVEX COSTS: IS IT WORTH MORE THAN A PENNY?

*Huw David Dixon**

ABSTRACT

In this paper we analyse the implications of integer pricing for Bertrand Edgeworth oligopoly with strictly convex costs. When price is a continuous variable, there is a generic non-existence of pure-strategy equilibrium. In the case of integer pricing, this is not so. We characterize a set of possible single price equilibria around the competitive price, which if non-empty will constitute the set of single price equilibria if the industry is large enough. Furthermore, we provide an example in which the highest equilibrium price can be arbitrarily far from the competitive price.

This paper considers the general Bertrand-Edgeworth model where firms set prices, producing an homogeneous product with strictly convex and differentiable cost functions. A continuum of perfectly informed consumers buy from the cheapest firm, the contingent demand for higher priced firms arising from the unsatisfied demand at lower prices. When prices are chosen from some closed interval $[0, \bar{P}]$, there is a well known result that in general no pure strategy equilibrium exists (see Edgeworth (1922), Dixon (1987a)), and if a pure strategy equilibrium does exist, then it must be the competitive price (Shubik (1958, p. 100)).

This paper considers the implications for pure strategy Bertrand-Edgeworth equilibria of restricting prices to setting integer prices, as in Maskin and Tirole (1988) and Harrington (1990). In those papers there is

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costless production, which yields the well known result that there are two pure-strategy equilibria: the traditional zero price Bertrand equilibrium, and also one where all firms set prices equal to 1 (or in general one 'penny' above unit cost). As Harrington (1989) has argued, the unit price equilibrium is preferable to the competitive equilibrium, primarily because it does not involve firms playing dominated strategies. However, one penny is not that much of a difference, and the standard Bertrand equilibrium with no profits may well be a good approximation. With strictly positive costs, the issues raised are both more important and interesting. The questions which the paper addresses are: does the restriction of firms strategies to a finite compact set solve the non-existence problem; if so, what is the possible set of equilibria? Will the possible equilibria be close to the competitive outcome in absolute terms, as in the 'one penny' equilibrium with constant costs?

The answers given to these questions are in three parts. First, integer pricing does solve the existence problem in the sense that we can characterize a set of prices such that any integer in that set will constitute a (single price) equilibrium if the industry is large enough (in the sense of replicating the demand and supply side). Secondly, whilst the set of equilibrium prices might include the competitive price, in general it will include integers *above* the competitive price. Restricting firms to setting integer prices makes the price-setting game less competitive, since it rules out ε -undercutting, where firms can undercut each other by an arbitrarily small ε and capture the whole market. In this model firms have to undercut by at least one unit. Thirdly, we answer the question 'is it worth more than a penny?'. We provide an example where we can characterize the equilibrium set, and show that integer prices *arbitrarily far from the competitive price* can be equilibria. However, whilst they can be arbitrarily far away in *absolute* terms, they are still close in *relative* terms. The equilibria we explore do matter by more than a penny, but not by significantly more.

The reason for looking at integer prices are both technical and plausible. With continuous prices, best-response functions are not defined in Bertrand-Edgeworth models (Maskin and Tirole (1988)). On the empirical level, integer pricing and minimum currency denominations make integer pricing more realistic. Treating prices as continuous is itself a convenient simplification, but one that unnecessarily precludes some prices as equilibria because they are prone to ε -undercutting.

1. THE MODEL

Without replication there are n firms. In an r -replica industry, there are rn firms $1 = 1, 2, \dots, rn$. Each firm sets a price p_i , the rn vector of prices being denoted P and the $rn - 1$ vector of prices excluding i being P_{-i} . Each firm

has the same cost function $c: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ giving total cost as a function of output:

Assumption A1: $c(\cdot)$ is strictly convex, continuously differentiable and strictly increasing.

On the demand side, without replication there is an industry demand function $F: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that:

Assumption a2: There exist bounded and strictly positive constants \bar{P} , $K > 0$ such that $F(p) = 0$ when $p \geq \bar{P}$, and $F(p) \leq K$. F is continuous, strictly decreasing and $p \cdot F(p)$ is weakly concave on $[0, \bar{P}]$.

In an r -replica industry, industry demand becomes $r \cdot F(p)$. From A1 we define the firm's profit and supply functions:

$$\zeta(p) = \max_x p \cdot x - c(x)$$

$$S(p) = d\zeta/dp$$

Under A1, S is continuous and strictly increasing when positive. We assume that $S(\bar{P}) > 0$, so that under A1-2 we can define the competitive price $\theta \in (0, \bar{P})$:

$$F(\theta) = n \cdot S(\theta) \tag{1}$$

Replication leaves θ unaffected, since both sides of (1) are multiplied by r .

In order to facilitate the analysis, we need to define some functions. Firstly $\chi(p) = p \cdot F(p) \cdot n^{-1} - c(F(p)/n)$. Under A1-2, since revenue is non-convex and costs are strictly convex, $\chi(p)$ is strictly concave, possessing a unique maximum at M , where $\theta < M < \bar{P}$. For interest's sake we assume that $M > \theta + 1$. Note that at θ :

$$\chi(\theta) = \zeta(\theta); d\zeta(\theta)/dp = d\chi(\theta)/dp$$

We define the price \hat{p} by the relation

$$\chi(\hat{p}) = \zeta(\hat{p} - 1)$$

Under A1-2, \hat{p} is uniquely defined, and for interest's sake we assume $\hat{p} < M$. \hat{p} is a real number, and is the price at which firms are indifferent between sharing demand with the other $n - 1$ firms (to earn $\chi(p)$) and undercutting by one unit to capture the whole industry demand (to earn $\zeta(p - 1)$), as depicted in Figure 1. Note that because of the concavity of demand and the strict convexity of cost functions, it will pay to undercut by one unit when $p > \hat{p}$, and not otherwise.

Let us now consider the specification of contingent demand, which gives the demand for firm i as a function of the m prices set, $d_i: \mathbb{R}_+^m \rightarrow \mathbb{R}_+$. The precise assumptions made about contingent demand can be very important in Bertrand-Edgeworth models. Here we will adopt the simplest form, variously called efficient/parallel rationing or compensated

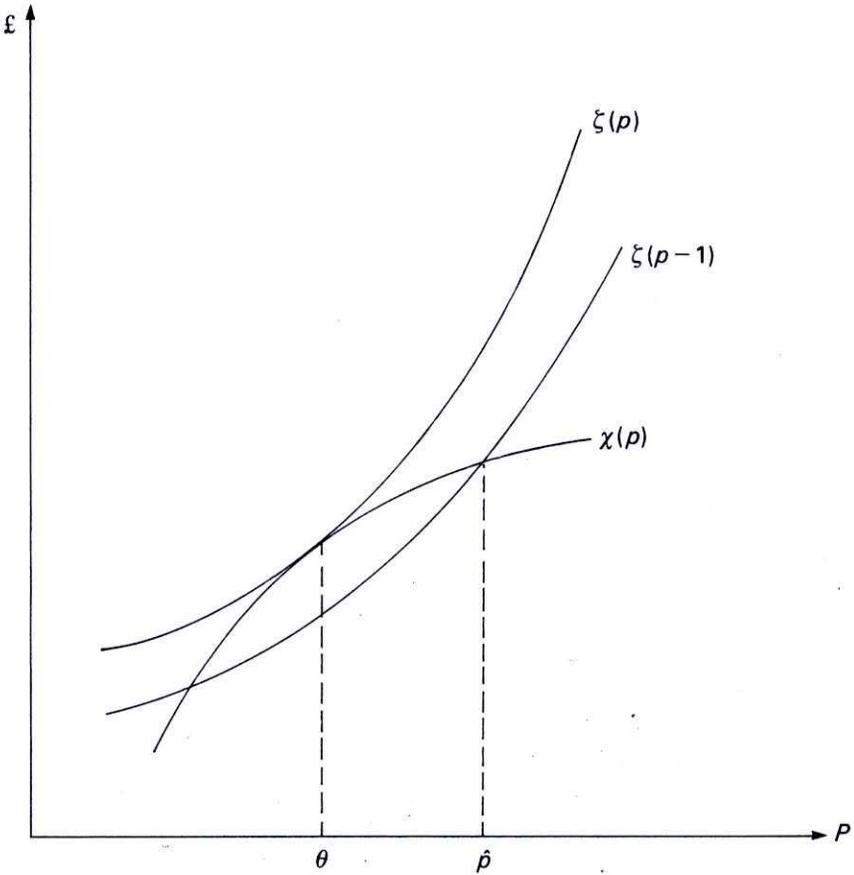


Fig. 1. The definition of \hat{p}

demand (see Dixon (1987b) for a detailed discussion). As a preliminary, let us define the function $n_i = \# \{j = 1, \dots, m: p_j = p_i\}$

Assumption A 3: Contingent Demand.

$$d_{ri}(p_i, p_{-i}) = \max \left[0, \frac{r \cdot F(p_i) - \sum_{p_k < p_i} S(p_k)}{n_i(P)} \right]$$

We are now in a position to define the firms payoff function Π_{ri} which gives profits as a function of prices P set:

Definition: Payoff Function. $\Pi: \mathbb{R}_+^m \rightarrow \mathbb{R}_+$

$$\Pi_{ri} = \begin{cases} \xi(p_i) & d_{ri}(P) \geq S(p_i) \\ p_i \cdot d_{ri}(P) - c(d_{ri}(P)) & d_{ri}(P) \leq S(p_i) \end{cases}$$

Π_{ri} reflects the standard Bertrand-Edgeworth voluntary trading constraint. If demand exceeds profit maximizing supply $S(p_i)$, the firm

turns away customers and earns $\zeta(p_i)$; otherwise output is demand determined.

2. EQUILIBRIUM

The main results of this paper concern the set of possible single-price equilibria (SPE) that exist in the game $[\mathbb{Z}_+, \Pi_{ri}: 1 + 1, \dots, m]$. In the game with continuous prices $[\mathbb{R}_+, \Pi_{ri}: i = 1, \dots, m]$, it is well known that no pure strategy equilibria exist (see Dixon (1987a)), although mixed-strategy equilibria exist (Dixon (1984), Maskin (1986)). It is worth recalling briefly why no pure strategy equilibria exist. If we restrict our attention to SPE, no price above θ can be an equilibrium, since it is prone to ε -undercutting. No price below θ can be an equilibrium, since there will be excess demand for output, and firms can profitably deviate by raising their price a little. Thus θ is the only possible SPE. However, at the competitive price, firms are on their supply functions with price equal to marginal cost. They will only be unable to profitably deviate from this if their demand curve is 'horizontal' at θ . However, if a firm raises its price slightly, it can meet the residual demand left unserved by the other firms who are producing $S(\theta)$. Since price equals marginal cost at θ , the loss in sales has no first order effect on profits, whilst the rise in price does, so that it pays the firm to raise its price. The only exception to this is where the industry demand is flat at θ , which is ruled out by A2.

In this paper, matters are less simple because firms can only change prices in integer units. We can however define a set of prices E , which contains the set of SPE if the economy is large enough. We have already defined \hat{p} as the price at which firms are indifferent between cutting price by one unit and sharing demand. Let us further define the largest integer at or below θ :

$$\hat{\theta} = \max \{ p \in \mathbb{Z}_+ : p \leq \theta \}$$

We can now define the set of integers E :

$$E = [\hat{\theta}, \hat{p}] \cap \mathbb{Z}_+$$

We can now state our main result:

Theorem (existence): A1-3. Consider $[\mathbb{Z}_+, \Pi_{ri}: i = 1, \dots, m]$.

There exists r' such that for $r > r'$,

(a) If $p \in E$, and $p \geq \theta$, then p is an SPE.

(b) If $p = \hat{\theta}$, then p is an SPE iff:

$$F(\hat{\theta} + 1) < n \cdot S(\hat{\theta})$$

(c) If $p \notin E$ then p is not an SPE.

The Theorem can be explained intuitively. If all firms set a price strictly greater than θ , there is an excess supply which increases without bound under replication. Although by under cutting a firm can increase its sales, it has to cut price by at least one unit. For prices at or below \hat{p} , this is not worth it. Replication also ensures that no firm will wish to raise its price either, since eventually there will be sufficient excess supply for the $m - 1$ other firms to meet all demand, leaving none for the price raiser. These arguments are perfectly general, and will hold for any specification of contingent demand, not just A3.

If we turn to the case of $\hat{\theta}$, we have in part (b) a necessary and sufficient condition for $\hat{\theta}$ to be an equilibrium (for r large enough). Note that if θ is itself an integer, then $\hat{\theta} = \theta$, and the condition is automatically met. If θ is not an integer, then $\hat{\theta} < \theta$ and the condition may or may not be met. The condition is represented in Figure 2: in 2a it is met; in 2b it is not. The condition is necessary and sufficient for the contingent demand of a firm raising its price above $\hat{\theta}$ to go to 0 as r tends to ∞ . Although there is excess demand at $\hat{\theta}$, which becomes unbounded with r , any firm wishing to raise its price must do so by at least one unit. The slope of the demand curve is also increasing with r . There is thus a knife-edge situation: if the inequality in (b) is met, then contingent demand tends to zero with r for integer prices above $\hat{\theta}$; if the inequality is violated, then contingent demand is at least $S(\hat{\theta})$, in which case a profitable price rise must be possible. Part (b) of the theorem is sensitive to the specification of contingent demand, and depends on A3. For other specifications $\hat{\theta}$ might never be an equilibrium.

Having examined SPE, what can we say about multiple price equilibria? With continuous prices, very simple arguments rule out multiple price equilibria (see e.g. Dixon (1987a)). With integer pricing the matter is a little more intricate, and much less clear. Whilst there can be no equilibria with more than two prices, we cannot rule out in special cases two-price equilibria. However, we can show that if the industry is large enough, then multiple price equilibria are ruled out, at least under contingent demand A3:

Proposition: A1–3. Let $P \in \mathbb{Z}_+^m$ be an equilibrium in $[\mathbb{Z}_+, \Pi_{r_i}; i = 1, \dots, n]$. For r large enough, P is an SPE.

Lastly, in the case of constant/zero unit cost, the highest SPE involves the firms setting the one-penny price. This equilibrium is close in absolute terms to θ whatever the demand conditions. A difference in price of one penny is not perhaps all that important, although it does wonders for the profit margin in the firm! In this paper, however, there is a range of possible equilibrium prices $[\hat{\theta}, \hat{p}]$. The extent to which the integer SPE can deviate from the competitive price θ will depend upon the precise cost and demand functions that define \hat{p} . We will provide an example where the *absolute* differences $\hat{p} - \theta$ can be arbitrarily large.

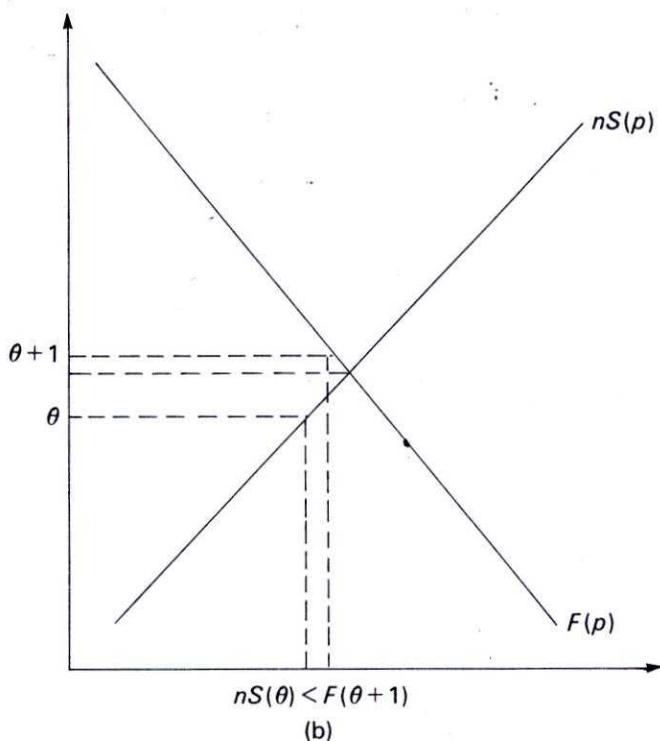
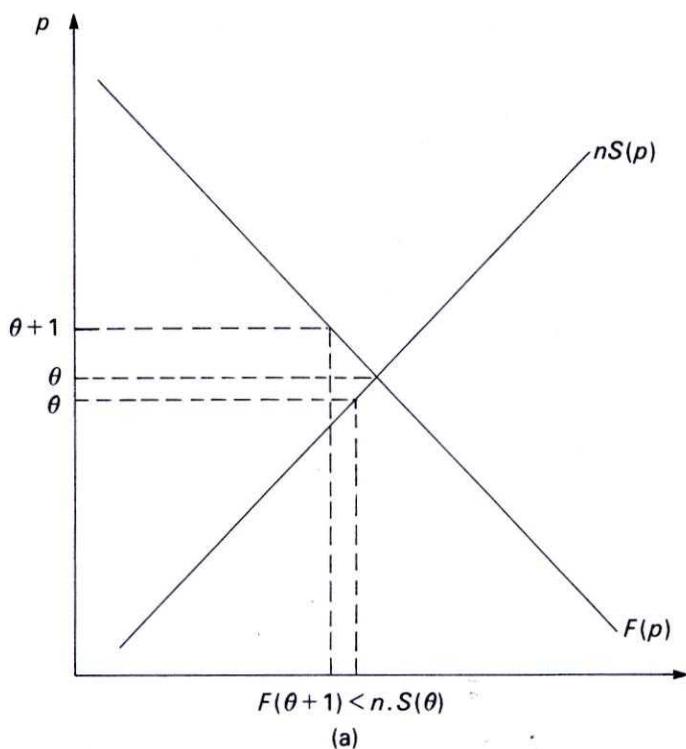


Fig. 2. Illustration of theorem part (b).

Example

$$F(p) = \bar{P} - p(\bar{P} < 1)$$

$$c(x) = x^2$$

$$\text{Hence: } \chi(p) = p \cdot (\bar{P} - p)(1/2) - (\bar{P} - p)^2(1/4)$$

$$\zeta(p - 1) = (p - 1)^2(1/4)$$

$$\theta = \bar{P}/2$$

Solving for \hat{p} by setting $\chi(p) = \zeta(p - 1)$ yields:

$$\hat{p} - \theta = 1/4 + (1 + 4(\bar{P} - 1))^{1/2}(1/4)$$

As can be seen, the absolute difference between \hat{p} and θ is unbounded as \bar{P} tends to infinity, and of order $\bar{P}^{0.5}$. Hence in this case, unlike the standard Bertrand model, integer pricing does make a difference of more than one penny. However, although the *absolute* difference $\hat{p} - \theta$ is unbounded, the difference *relative* to θ tends to 0:

$$\frac{\hat{p} - \theta}{\theta} = \frac{1}{2\bar{P}} [1 + [1 + 4(\bar{P} - 1)]^{0.5}]$$

Clearly, this 'markup' goes to zero as \bar{P} goes to infinity. Hence although integer pricing may be worth more than a penny in absolute terms, it need not be in relative terms.

A couple of comments are worth making on this example. First, we have considered what happens to the 'markup' of \hat{p} over θ as \bar{P} tends to infinity. An alternative would be to consider the convergence of the absolute difference $\hat{p} - \theta$ to zero when \bar{P} is constant and the money grid is made smaller (i.e. the size of monetary units on the real line goes to zero). Secondly, in the example we have considered, the welfare loss due to integer pricing will be small if we use a Harberger-Meade measure of lost consumer surplus (which is proportional to the square of the markup), at least relative to the total surplus earned in the competitive outcome (which will be increasing and unbound in \bar{P}).

CONCLUSION

This paper has taken the standard Bertrand-Edgeworth model with strictly convex costs and introduced the plausible assumption that prices are integers rather than real numbers. The results of this paper are important for two reasons:

(i) Integer pricing provides a solution to the problem of non-existence that is endemic to such model with continuous prices.

(ii) The equilibria that exist in this framework can be arbitrarily far away from the competitive price in absolute terms, although not proportionally.

On a broader point, the results of this paper indicate that the assumption of continuous prices is itself only an approximation, made for convenience. When continuity itself causes problems of non-existence, perhaps we should drop it.

APPENDIX: PROOFS

Theorem

(a) Let us define the minimum variation of F on $[\theta, \hat{p}]$ for integers p, q where $p \neq q$:

$$\Delta = \min_{p, q} |F(p) - F(q)|$$

Clearly, $\Delta > 0$, since F is strictly monotonic on $[\theta, \hat{p}]$.

From the definition of \hat{p} , no firm will want to undercut if all firms set a price p in $(\theta, \hat{p}]$. Furthermore, no firm will wish to raise its price if $d_{ri}(q, P_{-i}) = 0$ for all integers $q > p$. Under A3, if all firms other than i set price p , we have:

$$\begin{aligned} d_{ri}(q, p) &= \min [0, rF(q) - (rn - 1)S(p)] = 0 \\ &\Rightarrow rF(q) - (rn - 1)S(p) \leq 0 \end{aligned}$$

Since $S(p) > S(q)$, and $F(q)F(\theta) - \Delta$, we have:

$$\begin{aligned} rF(\theta) - r\Delta - (rn - 1)S(\theta) &\leq 0 & -r\Delta + S(\theta) &\leq 0 \\ r &\geq S(\theta)/\Delta \end{aligned}$$

Hence for r large enough, no firm can profitably defect from any SPE with $\theta < p \leq \hat{p}$.

(b) Consider the case of $\hat{\theta} \leq \theta$. If all firms set $p = \hat{\theta}$ then there will be excess-demand. $\hat{\theta}$ can only be an equilibrium if it does not pay an individual firm to raise price. If firm i raises its price to $q > \hat{\theta}$, its contingent demand will be:

$$d_{ri}(q, \hat{\theta}) = r.[F(q) - nS(\hat{\theta})] + S(\hat{\theta}) \leq r.[F(\hat{\theta} + 1) - nS(\hat{\theta})] + S(\hat{\theta})$$

Since $F(q) \leq F(\hat{\theta} + 1)$. If $F(\hat{\theta} + 1) < nS(\hat{\theta})$, then d_{ri} converges pointwise to 0 as r tends to infinity. In this case no firm will wish to raise its price. For necessity, if this strict inequality is not satisfied then $d_{ri}(\hat{\theta} + 1, \hat{\theta}) \geq nS(\hat{\theta})$, so that the firm can always increase its profits by raising its price by one unit and selling the same output. If $\hat{\theta} = \theta$, then by definition and the strict monotonicity of F and S the strict inequality is met.

(c) There can be no SPE above \hat{p} by definition; no price below $\hat{\theta}$, since a firm could raise its price to $\hat{\theta}$ and earn $\zeta(\hat{\theta})$. QED.

Proposition

We will prove this in two steps: step (a) shows that no three price (or more price) equilibria can exist: step (b) shows that for r large enough no two price equilibria exist.

(a) In a multiple price equilibrium (MPE), define the highest price P_h , and the lowest price set P_L . All firms k setting prices below P_h will be supply constrained, and earn $\zeta(p_k)$, since otherwise the highest price firms would have no demand and earn zero profits. Furthermore, it is easily shown that the prices set must be consecutive (otherwise the lowest priced firms could increase profits by setting a price in the "gap" or just below). Again, it is obvious that $P_h \geq \theta$, since otherwise any firm setting a lower price could raise its profits to $\zeta(P_h)$ by setting P_h (since there will still be excess demand). If we define the output of the highest priced firms as X_h , under A3 we have:

$$\sum_{p_i = P_h} X_h + \sum_{p_i = P_{h-1}} S(P_h - 1) + \sum_{p_i = P_L} S(P_L) \leq rF(P_H) \quad (A1)$$

The inequality will be an equality if there is no excess demand at P_h , in which case $X_h \leq S(P_h)$. Now consider a firm k setting the lowest price, and contemplating raising its price one penny to $P_h - 1$. Its contingent demand will be d_{rk} :

$$d_{rk} \geq r.F(P_h - 1) - \sum_{p_i \geq P_H} S(p_i) \quad (A2)$$

However, since F is monotonic, and $X_h > 0$ in (A1) it follows that

$$d_{rk} \geq S(P_L) \quad (A3)$$

That is, a low price firm can raise its price by one integer and sell at least the same amount, so that the initial position could not have been an equilibrium.

(b) Consider a two price equilibrium. then (A1) becomes:

$$\sum_{p_i = P_H} X_h + \sum_{p_i = P_L} S(P_L) = rF(P_H) \quad (A4)$$

where $P_H \geq \theta$, and $P_L = P_H - 1$. Furthermore, suppose that a proportion H_r of firms set the high price, $0 < H_r < 1$, then (A4) becomes:

$$H_r X_h + (1 - H_r) S(P_L) = F(P_H) \quad (A5)$$

Let us define the profits of the high price firm Π_h . We now derive a contradiction.

First, suppose that a low-price firm raises its price by one unit. Then the output of the higher priced firms will raise to X'_H , where:

$$X'_H = X_H + \frac{S(P_H) - X_H}{r.n.H_r + 1}$$

with a resultant increase in profits to $\Pi'_h > \Pi_h$. The firm raising its price will also earn Π'_h , and since the initial position is an equilibrium, it follows that:

$$S(P_L) \geq \Pi'_h > \Pi_h. \tag{A5}$$

Second, suppose that a higher priced firm *reduces* its price to P_L , then it must not earn more than Π_h , so that there is an upper bound on the excess demand for lower-priced firms:

$$r.F(P_L) - (1 - H_r).r.n.S(P_L) < S(P_L) \tag{A6}$$

If (A6) were violated, then from (A5) it must pay to deviate. Note that (A6) implies that $P_L > \theta$, since otherwise there would always be contingent demand of at least $S(\theta)$ irrespective of H_r . (A6) implies that there cannot be too many firms setting a high price (an upper bound on H_r), since otherwise there would not be enough low priced firms to dissuade the high-price firms from setting P_L . Since from (A5) $P_L \geq \theta$, we also have a lower bound H^* on H_r , since the contingent demand for high-price firms must be positive, $X_H > 0$. (A4). Hence:

$$H_r > H^* = \frac{n.S(P_L) - F(P_L)}{nS(P_L)} \tag{A7}$$

We now derive the contradiction. The total output of high-price firms is $r.n.H_r.X_H$. Under (A4, A7) and we have:

$$\begin{aligned} r.n.H_r.X_H &= r.F(P_H) - r.n.(1 - H_r).S(P_L) \\ &< r.F(P_L) - r.n.(1 - H^*).S(P_L) \end{aligned}$$

Hence

$$X_H < \frac{S(P_L)}{H^*.r.n}$$

Clearly, (since P_L is bounded by \bar{P}), as $r \rightarrow \infty$, $\text{Limsup } X_H = \text{Limsup } \Pi_h = 0$. But $\zeta(\hat{\theta}) > 0$, so that for r large enough $\Pi_h < \zeta(\hat{\theta})$, and the initial position cannot be an equilibrium, the desired contradiction. QED

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