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Linear-homothetic preferences

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Abstract

This paper develops a new class of linear homothetic (*LH*) preferences that result in Marshallian demands that are linear in price under the assumption that agents take aggregate price indexes as given (as in monopolistic competition). The preferences are represented by a cost-function that has one parameter, which can be interpreted as the elasticity of demand when all prices are equal. The cost-function has a restricted form that allows the elasticity of demand to be compatible with any number of commodities. © 2000 Elsevier Science S.A. All rights reserved.

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1. Introduction

This paper introduces a new class of homothetic preferences which have the useful property that the demand for each commodity is in an important sense linear in own price. *Linear Homothetic (LH)* preferences will be useful in general equilibrium settings, particularly with wage or price-setting agents. Linearity is a very useful property for demand curves: it has been very extensively analyzed in the partial equilibrium setting in a variety of contexts (for example union-firm bargaining, industrial organization) where it is the reference point for many modelling exercises. In a general equilibrium context, however, linearity of demand is presently modelled with quadratic preferences¹. The use of quadratic preferences has properties that may be undesirable in some contexts: in particular, there is a satiation point. Homotheticity is also widely used in applied general equilibrium models, including

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¹For example, by specifying preferences for two goods x and y as quasi-linear $x + u(y)$, with the further assumption that $u(y) = y - y^2$.

trade theory, macroeconomics and CGE. The two most commonly used homothetic preferences are CES and Cobb–Douglas (CD). Whilst appropriate in some applications, they have special properties that may be undesirable in others: for example, CES and CD give rise to non-monotonic reaction functions in Cournot oligopoly which can make asymmetric equilibria difficult to analyze².

The approach we adopt is to specify a particular functional form for the cost function, which has one parameter — the elasticity of demand when prices are equal. Under the assumption of *Aggregate Price Taking (APT)*, as in the Dixit and Stiglitz (1977) model of monopolistic competition, this results in a linear Marshallian demand curve. The cost function is constructed in such a way that preferences are homothetic. We develop a restricted version of our cost-function that allows for any combination of equilibrium elasticity and number of industries/goods while simultaneously satisfying all the required properties.

2. The LH cost function

There are n products, $i = 1 \dots n$, the n -vector of prices p_i is \mathbf{p} . The expenditure function takes the homothetic form $E(\mathbf{p}, u) = b(\mathbf{p}) \cdot u$ where $b: \mathfrak{R}_+^n \rightarrow \mathfrak{R}_+$ is the unit cost of utility. b is a unit cost function if it satisfies the following properties³: (i) b is non-negative and non-decreasing in prices; (ii) b is homogenous of degree 1 and concave in \mathbf{p} ; (iii) b is continuously differentiable⁴.

Consider the function b

$$b(\mathbf{p}) = [\mu + \gamma(\mu - \pi)] \quad (1)$$

where $\gamma \geq 0$, and $\mu = 1/n \sum_{i=1}^n p_i$; $\pi = (1/n \sum_{i=1}^n p_i^2)^{1/2}$. If all prices are equal ($p_i = p$) then $\mu = \pi = p$; otherwise⁵ $b \leq \mu < \pi$.

The function b clearly satisfies properties (ii) and (iii) for any $\mathbf{p} \in \mathfrak{R}_+^n$: however, property (i) need not be satisfied for all $\mathbf{p} \in \mathfrak{R}_+^n$. Denoting db/dp_i by b_i , Proposition 1 gives a necessary and sufficient for property (i) to hold for all prices.

Proposition 1. $b \geq 0$ and $b_i \geq 0$ for all $\mathbf{p} \in \mathfrak{R}_+^n$ iff

$$\gamma \leq \frac{1}{\sqrt{n} - 1} \quad (2)$$

Proof. A necessary and sufficient condition for $b \geq 0$ is $\gamma/(1 + \gamma) \leq \mu/\pi$. From the definitions of μ and π the minimum value of μ/π is achieved when all prices but one are zero, i.e. $\min \mu/\pi = 1/\sqrt{n}$. Hence a necessary and sufficient condition for $b \geq 0$ for all prices is

²See for example Santoni (1996) for an application of union-firm bargaining in an oligopoly with Cobb–Douglas preferences.

³For a fuller listing of necessary and sufficient conditions, see (Diewert, 1982, pp. 537–547).

⁴Continuous differentiability is not required for b to be a cost-function: however this enables us to apply Shephard's lemma.

⁵ π is a strictly convex function. For a given value of μ it is minimized when all prices are equal.

$$\frac{1}{\sqrt{n}} \geq \frac{\gamma}{1 + \gamma} \tag{3}$$

A necessary and sufficient condition for $b_i \geq 0$ is $(1 + \gamma)/\gamma \geq p_i/\pi$. The maximum value of p_i/π is achieved when all prices $j \neq i$ are zero, i.e. $\max p_i/\pi = \sqrt{n}$. Hence a necessary and sufficient condition for $b_i \geq 0$ for all prices is

$$\sqrt{n} \leq \frac{1 + \gamma}{\gamma} \tag{4}$$

Both (3), (4) are equivalent to (2). This is a sufficient condition for $b \geq 0$ and $b_i \geq 0$: it is necessary since these properties are to hold for all prices $\mathbf{p} \in \mathfrak{R}_+^n$. \square

When $\gamma = 0$, condition (2) is satisfied for all prices: b then represents Leontief preferences. We derive below a more general form of the cost function which we call the *restricted LH* cost function which ensures that the non-negativity conditions are satisfied for all γ and n . First, however, we assume that (2) is satisfied, so that the budget shares α_i corresponding to (1) are given by Shephard’s lemma:

$$\frac{p_i x_i}{Y} = \frac{p_i}{b} b_i \equiv \alpha_i(\mathbf{p}) \tag{5}$$

where Y is total expenditure on all commodities yielding the Marshallian demand

$$x_i = \left[(1 + \gamma) - \gamma \frac{p_i}{\pi} \right] \frac{Y}{nb} \tag{6}$$

This is non-linear, since π is non-linear. However, we assume in the spirit of the Dixit–Stiglitz ($D-S$) model of monopolistic competition (Dixit and Stiglitz, 1977) that there is *Aggregate price taking* (APT): The aggregate price indices π and b are treated as parameters by individual agents⁶. Under the assumption of Aggregate Price Taking (APT), (6) is perceived linear in p_i . Note that the linear demand has a *choke-off price*: $x_i \geq 0$ only when $p_i/\pi \leq (1 + \gamma)/\gamma$. This is exactly the same as the condition for non-negative own price derivatives as stated in Proposition 1, since the quantities are derived using Shephard’s Lemma. Treating π and b as given, the own price elasticity of demand (in absolute value) is

$$\varepsilon_i(\mathbf{p}) = \frac{\gamma \cdot (p_i/\pi)}{1 + \gamma - \gamma \cdot (p_i/\pi)}$$

In the case of a symmetric equilibrium (i.e. each sector is the same), $p_i = \pi$, so that $\varepsilon = \gamma$. Hence the parameter γ is the elasticity of demand in *any* symmetric equilibrium. Note that the elasticity of demand is an *increasing* function of p_i if we treat π as constant.

⁶In fact, since $b - (1 + \gamma)\mu + \gamma\pi = 0$, treating any two of $\{b, \pi, \mu\}$ as given means the third is also given.

3. The restricted LH cost function

The previous analysis has been made under the assumption that given n the elasticity of demand satisfies condition (2). In the case of price-making agents (e.g. monopolistic firms) it is often necessary to assume that the industry demand is elastic: in this case (2) is overly restrictive⁷. We restrict the cost function b in such a way that the properties (particularly property (i)) of a cost function are satisfied for any γ and n . First, let us define the subset of \mathfrak{R}_+^n where $b_i \geq 0$ given γ

$$\beta(\gamma, n) = \{\mathbf{p} \in \mathfrak{R}_+^n : b_i(\mathbf{p}) \geq 0 \text{ for all } i = 1 \dots n\}$$

Given n and $\gamma \geq 0$, $\beta(\gamma, n) \subseteq \mathfrak{R}_+^n$ is non-empty⁸. Outside $\beta(\gamma, n)$, the function is adjusted so that all budget shares are non-negative. This is achieved by sequentially lowering prices starting with the highest until there are no non-negative budget shares using a *price-capping function* $g: \mathfrak{R}_+^{n+1} \rightarrow \mathfrak{R}_+^n$, which places a cap (upper bound) of λ on prices, yielding the n -vector of capped prices $\hat{\mathbf{p}}$

$$\hat{\mathbf{p}} = g(\lambda, \mathbf{p}) = (\min[\lambda, p_i])_{i=1 \dots n}$$

Clearly, g is homogeneous of degree 1 and continuous in (\mathbf{p}, λ) . However, g is not continuously differentiable: We have⁹

$$\frac{d\hat{p}_i}{dp_i} = \begin{cases} 1 & \text{if } p_i \leq \lambda \\ 0 & \text{if } p_i \geq \lambda \end{cases} \forall i; \quad \frac{d\hat{p}_i}{dp_j} = 0 \ (i \neq j); \quad \frac{d\hat{p}_i}{d\lambda} = \begin{cases} 1 & \text{if } p_i \leq \lambda \\ 0 & \text{if } p_i \geq \lambda \end{cases} \forall i. \tag{7}$$

Given \mathbf{p} , define the maximum price $p^{\max} = \max\{p_i\}$. There are two regimes: when $\lambda = p^{\max}$, the price cap is inoperative (i.e. $\mathbf{p} = \hat{\mathbf{p}} = g(\lambda, \mathbf{p})$); when $\lambda < p^{\max}$ the price cap is operative. Without loss of generality, we will assume that the price vector is ordered in terms of uncapped price, $p_1 \leq p_2 \dots p_{n-1} \leq p_n$, so that $p^{\max} = p_n$. If the price cap is operative, then there exists j such that $p_{j-1} < \lambda \leq p_j$, so that $\hat{\mathbf{p}} = (p_1, p_2 \dots p_{j-1}, \lambda, \dots \lambda)$. We can then define the function $\lambda: \mathfrak{R}_+^n \rightarrow \mathfrak{R}_+$: which gives the largest value of λ such that the capped price vector is in $\beta(\gamma, n)$, i.e.

$$\lambda(\mathbf{p}) = \max_{p^{\max} \geq \lambda \geq 0} \lambda \tag{8}$$

$$s.t. \ b_i(g(\lambda, \mathbf{p})) \geq 0 \quad i = 1 \dots n$$

By the theorem of the maximum, $\lambda(\mathbf{p})$ is continuous in \mathbf{p} (since b_i and g are continuous and λ is chosen from a compact set). When $\lambda(\mathbf{p}) < p^{\max}$, it implies that $b_n < 0$ at the uncapped prices. With capping binding at p_j , from the definition of $\lambda(\mathbf{p})$ we have $b_k(\hat{\mathbf{p}}) = 0$ for $k \geq j$, and $b_k(\hat{\mathbf{p}}) > 0$ for $k < j$. The following properties are useful.

⁷To justify *APT*, a large number of industries is needed. From Proposition 1 for large n the elasticity has to be very small, tending to zero as $n \rightarrow \infty$.

⁸If all prices are equal $p_i = p$ for all i , then $b_i(\mathbf{p}) = 1/n > 0$. Hence $\beta(\gamma, n)$ has at least one element and is non-empty. Note that $\beta(\gamma, n)$ is also convex and closed.

⁹Note that the *RHS* and *LHS* derivatives of g are not equal when $p_i = \lambda$.

Lemma. Derivatives of $\lambda(\mathbf{p})$ ¹⁰: (i) $\lambda_i = 0$ if $p_i > \lambda$; (ii) $1 > \lambda_i > 0$ if $p_i < \lambda$ and $\lambda < p^{\max}$; (iii) $\lambda_i = 0$ if $p_i < p^{\max}$ and $\lambda = p^{\max}$.

Proof. First consider (i) and (ii) where price capping is operative. With capping binding at p_j , we have $b_j(\hat{\mathbf{p}}) = 0$: totally differentiating this with respect to (λ, p_i) yields

$$\frac{d\lambda}{dp_i} = \lambda_i = -b_{ji} \frac{d\hat{p}_i/dp_i}{\sum_{k=j}^n b_{jk}} \tag{9}$$

Since $\hat{p}_j = \lambda$, from (1),

$$b_{jk} = \frac{\gamma\lambda^2}{n^2\pi^3} \quad k > j, \quad b_{jj} = \frac{\gamma\lambda^2}{n^2\pi^3} - \frac{\gamma}{n\pi}$$

Hence

$$\sum_{k=j}^n b_{jk} = -\frac{\gamma}{n\pi} \left[1 - \frac{\lambda^2}{\pi^2} \frac{(n+1-j)}{n} \right] = -\frac{\gamma}{n^2\pi^3} \sum_{s=1}^{j-1} (p_s)^2 < 0$$

(i) If $p_i > \lambda$ then from (7), the numerator of (9) is 0, hence $\lambda_i = 0$.
 (ii) If $p_i < \lambda$ then from (7) the numerator of (9) is b_{ji} . Since $b_{ji} < \sum_{k=j}^n b_{jk}$, we have $0 < \lambda_i < 1$.
 Next consider (iii). Since $\lambda = p^{\max}$ the price cap is inoperative. If $p_i < \lambda = p^{\max}$ then from (8) $b_n(g(\lambda, \mathbf{p})) \geq 0$ for firm n ($p_n = p^{\max}$). Since $b_{in} > 0$, this will still hold if p_i increases slightly. Hence, $\lambda_i = 0$. \square

We can now define the *restricted LH cost function* $B(\mathbf{p})$

$$B(\mathbf{p}) = b(g(\lambda(\mathbf{p}), \mathbf{p})) \tag{10}$$

Proposition 2. Properties of restricted LH cost function $B(\mathbf{p})$.

- (i) B is continuous and homogeneous of degree 1 in \mathbf{p}
- (ii) B_i is continuous in \mathbf{p} , with $B_i(\mathbf{p}) = b_i(g(\lambda, \mathbf{p}))$
- (iii) B is concave in \mathbf{p}

Proof. (i) This follows since λ, b , and g are all continuous and linearly homogeneous in \mathbf{p} .
 (ii) Continuous differentiability of B follows from the continuous differentiability of b . From (10), we have

$$B_i = b_i + \sum_{p_j \geq \lambda} b_j \lambda_i$$

¹⁰When $\lambda = p_i$ and $\lambda < p^{\max}$, the LHS and RHS derivatives of λ are not equal: the LHS derivative is given by (ii), the RHS by (i).

However, from the definition of $\lambda(\mathbf{p})$, if $p^{\max} > p_j \geq \lambda$ then $b_j = 0$; if $p^{\max} = \lambda$ then $\lambda_i = 0$. Hence the *RHS* summation is zero (since λ_i bounded)¹¹. Since b_i is continuous in λ and \mathbf{p} , and λ is continuous in \mathbf{p} , the result follows.

(iii) Concavity of b implies that for any \mathbf{p}, \mathbf{p}'

$$b(\mathbf{p}) \leq b(\mathbf{p}') + \sum_{i=1 \dots n} b_i(\mathbf{p})(p_i - p'_i) \quad (\text{a})$$

Now suppose that $B(\mathbf{p})$ is not concave. Then there exists some \mathbf{p}, \mathbf{p}' such that

$$B(\mathbf{p}) > B(\mathbf{p}') + \sum_{i=1 \dots n} B_i(\mathbf{p})(p_i - p'_i)$$

From the definition of B and (ii) we have for $\hat{\mathbf{p}} = g(\lambda(\mathbf{p}), \mathbf{p})$ and $\hat{\mathbf{p}}' = g(\lambda(\mathbf{p}'), \mathbf{p}')$

$$b(\hat{\mathbf{p}}) > b(\hat{\mathbf{p}}') + \sum_{i=1 \dots n} b_i(\hat{\mathbf{p}})(\hat{p}_i - \hat{p}'_i)$$

But this contradicts (a). Hence B is concave in \mathbf{p} . \square

The restricted *LH* cost-function B allows any combination of (γ, n) while simultaneously satisfying all the desired properties of a cost function. However, in practice, it will be possible to use the simple cost-function b even when $\gamma > (\sqrt{n} - 1)^{-1}$. This is because in many applications, the actual prices set in a particular model will be ones resulting in positive outputs, where the price-cap in the restricted cost function is non-binding: e.g. monopolists will usually set prices which result in strictly positive outputs. However, Proposition 2 allows use to use b with confidence.

4. Conclusion

In this paper we have developed a new class of preferences that combines the desirable properties of homotheticity with (perceived) linearity. The class of preferences is parameterized by the elasticity of the Marshallian demand. We are able to develop this idea using the dual approach of the cost function. We believe that *LH* preferences will be useful in a variety of applications. Most importantly, in models where there is a within industry asymmetry in oligopoly: for example, where one firm moves first and/or firms have different cost structures.

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¹¹The non-differentiability of λ when $p_i = \lambda$ is not a problem: both the *LHS* and *RHS* derivatives exist and are bounded.

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