The Economic Journal, **102** (*March* 1992), 301–309
Printed in Great Britain

THE COMPETITIVE OUTCOME AS THE EQUILIBRIUM IN AN EDGEWORTHIAN PRICE–QUANTITY MODEL*

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In his model of price-setting duopoly (1897, 1922), Edgeworth envisaged a market where firms set prices, and trade occurs between households and firms given those prices. Edgeworth assumed trade was governed by a particular voluntary-trading constraint: that the output of any firm is the lesser of its demand and its 'supply', where the latter is interpreted as the profit-maximising output at the price set (this is particularly clear in his 1922 review where he considers firms with quadratic costs).

As is well known, except in the 'Bertrand' case of constant (zero) average/marginal costs, there are severe existence problems in Edgeworth's framework. With homogeneous outputs and zero cost up to capacity, for an intermediate range of capacities there will exist no pure-strategy Nash equilibrium, only cycles (Edgeworth, 1897), or mixed strategies (Bekman, 1965; Levitan and Shubik, 1972). With strictly convex costs, there is generic non-existence of pure-strategy equilibria (Dixon, 1987*a*), although mixed-strategy equilibria may exist (Dixon, 1984; Maskin, 1986), and ϵ-equilibria in a large industry (Dixon, 1987*a*). This non-existence is not solely due to the assumption of a homogeneous product which gives rise to discontinuities in firms' demands and payoff-functions. Even with differentiated products, so that demands are continuous in prices, the voluntary-trading condition will lead to non-existence if demand is elastic enough (Benassy (1989), theorem 3).

We propose a modification and generalisation of Edgeworth's model, which yields the competitive outcome as the only possible equilibrium, and most importantly gives a fairly weak sufficiency condition for existence of a pure-strategy equilibrium. We consider the most problematic and interesting case where firms have strictly convex costs, and produce a homogeneous good. The modification of Edgeworth's model is to allow firms $i = 1 \dots n$ to choose not only price P_i, but also q_i *the quantity they are willing to sell at that price*. the firm's strategy thus consists of an offer to trade up to a particular amount q_i at price P_i, with actual output being the minimum of demand and q_i. The Edgeworthian model is a special case of this, since it requires that q_i be the profit maximising output at P_i.

The price–quantity game in this paper is in many ways similar to the supply–function equilibria of Grossman (1981) (see also Klemperer and Meyer, 1989). The key difference is that whereas in Grossman's model firms choose a *function* relating price and quantity, here firms choose a *point* (P_i, q_i);

* I would like to thank seminar participants at Southampton, Warwick, and at the Royal Economic Society Conference 1991 (Warwick) for their comments.

also in this model firms are able to directly set their own price, unlike in supply–function equilibria where price is determined by a market-clearing condition. The model presented also differs from price–output games (Shubik, 1959, ch. 4; Boyer and Moreaux, 1987; Friedman, 1988), where firms choose output produced and prices (sales then being the minimum of output and demand). The key difference is that in these papers production costs are incurred at the chosen output irrespective of actual sales. No pure-strategy equilibrium exists in these models. Perhaps the closest approach to this paper is in the strategic-market game literature (e.g. Simon, 1984, and in particular Dubey, 1982). Apart from the fact that Dubey considers an exchange economy, the crucial difference is that in his model agents do not directly set prices, as in this paper.[1]

The model presented provides an account of how price-setting firms behave as if they were price-takers, and set the competitive price. Whilst it has been known for some time that the competitive outcome is the only possible equilibrium in pure strategies in an Edgeworth duopoly (Shubik, 1959, p. 101), non-existence is endemic. The advantage of the price–quantity game presented in this paper is that not only is the competitive outcome the only possible equilibrium, but also that existence is guaranteed under fairly weak conditions. As we show, equilibrium exists in the classic case of duopoly with quadratic costs. Existence rests on the idea that in equilibrium firms are willing to offer to trade beyond their profit-maximising output at the competitive price. This offer to trade will not be called upon (since if all firms set the competitive price, they will sell their competitive output). However, the offer to trade beyond their competitive output may serve to deter firms from deviating from the competitive outcome by raising their price, since in this event the remaining firms will raise output and hence reduce the deviant's demand and profits. In Section I we outline the model, in Section II we characterise the equilibrium, and provide the example of duopoly with quadratic costs.

I. THE MODEL

There are n firms $i = 1, \dots, n$, who each set price $p_i \in [0, \infty)$, and vector of all prices being $\mathbf{P} \in R^n_+$, and $\mathbf{P}_{-i} \in R^{n-1}_+$, the $n-1$ vector of all prices except P_i. Each firm i has the same cost function giving total cost as a function of output X_i.

Assumption 1: costs

All firms $i = 1 \dots n$ have the cost function $c(x_i)$, which is strictly increasing, strictly convex, and continuously differentiable.

Given Assumption 1, we can define the standard profit and supply functions:

$$S(p) = \arg\max p \cdot x - c(x)$$
$$\xi(p) = \max p \cdot x - c(x).$$

[1] In Dubey, agents make an offer to trade of the form: 'if the price of commodity j is P_j or less, I am willing to buy (sell) up to q_j' (1982, p. 113). The eventual equilibrium price is determined so as to make the offers to trade on both sides of the market consistent. Similarly, in Simon (1984), the market price is determined by 'arbitrageurs', who exploit the differences between bid and ask prices on both sides of the market. In this paper, agents on one side of the market (firms) set their own price directly.

The demand side is described by an atomless probability space (H, T, μ) of consumers $h \in H$. There is an integrable function $y: H \to R_+$ which assigns to each consumer an income. The total income of any subset of consumers $\hbar \in T$ is given by $Y: T \to R_+$, where

$$Y(\hbar) = \int_{\hbar} y(h)\, d\mu(h),$$

where we normalise $Y(H) = 1$. We assume that households have homothetic preferences, so that (for any price) demand is proportional to income. Implicitly holding prices in other markets constant, each consumer's demand is given by:

$$b(p) \,.\, y(h) \tag{1}$$

We make the following assumptions about $b(.)$:

Assumption 2

$b: R_+ \to R_+$. There exist bounded K, $\bar{P} > 0$ such that $b(P) = 0$ for $P \geqslant \bar{P}$, and $K \geqslant b(P) > 0$ when $P < \bar{P}$. $b(.)$ is strictly decreasing on $[0, \bar{P})$.

Although consumers may differ in income, they have common preferences. Due to homotheticity, the total demand of a particular subset $\hbar \in T$ of consumers depends on their total income, and not its distribution.

Having outlined the basic assumptions, we will specify how the market works. We extend the standard Bertrand–Edgeworth framework to allow each firm i to specify not only price P_i, but also q_i, *the maximum quantity which it is willing to sell* at that price. Firms simultaneously choose (P_i, q_i), and q_i is taken to be a credible pre-commitment to produce up to q_i: firm i can be obliged to sell up to q_i through some explicit legally binding contract or implicit loss of reputation. In the traditional Bertrand–Edgeworth equilibrium (Edgeworth, 1922; Shubik, 1959; Dixon, 1987*a*), q_i is not chosen: rather, q_i is the profit-maximising output at P_i, i.e. $q_i = S(P_i)$. In the Chamberlinian model where firms always meet demand, q_i is assumed to be large or unbounded.[2]

What limits the ability of firms to choose how much they offer to sell? Following the supply–function approach of Grossman (1981, p. 1157 (12*b*)), it seems reasonable to assume that no firm can be forced to sell an output that will make it go bankrupt. Therefore, we define the bankruptcy supply–function σ: $[0, \bar{P}] \to R_+$ where:

$$\sigma(P) = \max\{x \geqslant 0 : P \cdot x - c(x) \geqslant 0\}.$$

σ is strictly increasing and everywhere bounded by $\sigma(\bar{P})$. *Non-bankruptcy* implies that for a strategy (P_i, q_i) to be admissible, $q_i \leqslant \sigma(P_i)$. It should be noted that none of the results of the paper rests on q_i being restricted in this way: they would be valid if we allowed any $q_i \geqslant 0$. Rather, we impose non-bankruptcy to show that the existence result of the paper does not require any violation of this plausible condition. Firm i's compact convex strategy set is A_i:

$$A_i = \{(P_i, q_i) : P_i \in [0, \bar{P}], \quad q_i \in [0, \sigma(P_i)]\}$$

[2] See Benassy (1989) and Dixon (1990) for a discussion and comparison of Chamberlinian and Edgeworthian approaches to the issue of firms meeting demand.

where w.l.o.g. we truncate prices to $P_i \leqslant \bar{P}$. The n firms' strategies are represented by the two n-vectors $(\mathbf{P}, \mathbf{q}) \in A$, A being the Cartesian product of the A_is, a subset of R_+^{2n}.

We are now in a position to specify the contingent demand functions and trading process, to give each firm's demand, output and profit as a function of $(\mathbf{P}, \mathbf{q})$. As a preliminary, let us define the function $Y^*: [0, \bar{P}] \rightarrow [0, 1]$, where

$$Y^*(P) = \min [1, S(p)/b(P)].$$

Y^* gives the total income needed by any subset h of consumers in order for them to wish to buy $S(P)$ units of output at price P. Since total consumer income is 1, we use this to bound Y^*. Now, if a particular firm k sets a price P_k and sells all of its offered output q_k, the total income of the consumers who purchase its output is $(q_k/s(P_k)\, Y^*(P_k)$.

Given the prices $\mathbf{P}$ set by firms, the number of firms setting the same price as firm i is given by

$$n_i(\mathbf{P}) = \# \{k = 1, \dots n : P_k = P_i\}.$$

Clearly, $n_i(\mathbf{P}) = 1$ almost everywhere, and is discontinuous when $n_i(\mathbf{P}) > 1$.

The rationing mechanism is as in Edgeworth (1897, 1922): consumers buy from the lowest-priced firm they are able; if there is excess demand for a lower-priced firm, customers are rationed on the basis of first-come-first-served, which may be random or deterministic (see Dixon 1987*b*)). The contingent demand for higher-priced firms consists of those turned away by the lower-priced firms, and depends only on their total income. Since individual customers are of measure zero, the contingent demand will be deterministic irrespective of whether the rationing is random or deterministic. Formally, the firm's contingent demand function $d_i(\mathbf{P}, \mathbf{q})$ gives demand for firm i as a function of all firms' prices and offers.

Assumption 3 Contingent Demand $d_i : A \rightarrow R_+$

$$d_i(\mathbf{P}, \mathbf{q}) = \frac{b(P_i)}{n_i(\mathbf{P})} \max \left\{0, 1 - \sum_{P_k < P_i} [q_k/s(P_k)]\, Y^*(P_k)\right\}.$$

In effect, Assumption 3 implies that the demand for a firm depends upon the *proportion* of industry demand (household income) left over by lower-priced firms. If firms set the same price, they share demand, so that contingent demand is discontinuous.

The payoff function of the game gives profits earned as a function of $(\mathbf{P}, \mathbf{q}) \in A$:

Definition 1: payoff function $\Pi_i : A \rightarrow R_+$

$$\Pi_i(\mathbf{P}, \mathbf{q}) = \begin{cases} P_i \,.\, d_i(\mathbf{P}, \mathbf{q}) - c(d_i \mathbf{P}, \mathbf{q}) & d_i \leqslant q_i \\ P_i q_i - c(q_i) & d_i > q_i. \end{cases}$$

The fact that firms can offer to sell up to q_i means that in effect we have a 'voluntary' trading constraint which gives total output produced and sold as

$x_i = \min(q_i, d_i)$. In the standard Bertrand–Edgeworth model, we have $q_i = S(P_i)$, and there exists no pure-strategy equilibrium (under Assumptions 1–3), but a mixed-strategy equilibrium does exist (Dixon, 1984; Maskin, 1986). Note that the payoff function in this P–q game is discontinuous in prices, and that best-response functions are not always defined, as in the standard model.

II. EQUILIBRIUM

We will show in Theorem 1 that all pure-strategy equilibria in the modified Bertrand–Edgeworth model $(A_i, \Pi_i : i = 1 \ldots n)$ will involve the firms setting the competitive price θ and producing the competitive output $S(\theta)$, and we give sufficient conditions for existence. We will define the competitive price in two equivalent ways.

Definition 2: The Competitive Price θ

(*a*) $n . S(\theta) = b(\theta) . 1$. (*b*) $n . Y^*(\theta) = 1$.

We assume $S(\bar{P}) > 0$ to rule out zero production competitive outcomes, so that $\bar{P} > \theta > 0$. Definition 2(*a*) is standard; 2(*b*) states that total household income (RHS) is sufficient to just purchase the supply $n . S$ of firms at θ(LHS). The strict monotonicity properties of b, S and Y^* ensure that θ is unique, and that there is excess supply (demand) as P exceeds (is lower than) θ.

The crucial difference between this model and the standard Bertrand–Edgeworth model is that we have extended firms' strategies to allow them to choose $q_i \neq s_i(P_i)$. As a preliminary step, Lemma 1 determines when firms will take advantage of this opportunity.

Lemma 1. Let $(\mathbf{P}, \mathbf{q})$ be a pure-strategy Nash equilibrium in $[A_i, \Pi_i : i = 1 \ldots n]$. Then:

(*a*) $q_i > S(P_i)$ only if $d_i(\mathbf{P}, \mathbf{q}) \leqslant S(P_i)$.
(*b*) $q_i < S(P_i)$ only if $d_i(\mathbf{P}, \mathbf{q}) \leqslant q_i$.

Proof. (*a*) Assume the contrary to derive a contradiction, so that $q_i > S(P_i)$ when $d_i(\mathbf{P}, \mathbf{q}) > S(P_i)$. In this case, output is given by $x_i = \min(d_i, q_i) > S(P_i)$, so that:

$$\Pi_i(\mathbf{P}, \mathbf{q}) = P_i . x_i - c(x_i) < \zeta(P_i).$$

But then firm i can deviate from $(\mathbf{P}, \mathbf{q})$ by choosing $q^1 = S(P_i)$ so that:

$$\Pi_i^1(\mathbf{P}, q^1, q_{-i}) = P_i S(P_i) - c[S(P_i)] = \zeta(P_i)$$

hence increasing its profits, the desired contradiction.

(*b*) Similar proof. QED.

Lemma 1 establishes that no firm will wish to offer to sell more than $S(P_i)$ in equilibrium if demand exceeds $S(P_i)$, since otherwise it could offer to sell less and increase its profits. Conversely, if it does offer to sell more than $S(P_i)$, demand must be less than $S(P_i)$. Furthermore, a firm will only offer to sell less than $S(P_i)$ if $d_i \leqslant q_i$, otherwise it could increase profits by choosing a larger q_i.

Thus Lemma 1 demonstrates that firms will only offer a q_i different to $S(P_i)$ if their offer is not binding in equilibrium. However, whilst the offer may not be taken up in equilibrium, it may influence payoffs off the equilibrium path, and hence support an equilibrium.

Lemma 2 establishes that there can be no multiple-price equilibria, nor any single-price equilibria (SPE) at any price other than θ. The arguments are very similar to those employed in standard Bertrand–Edgeworth models and are in an appendix.

Lemma 2. Let $(\mathbf{P}, \mathbf{q})$ be a pure-strategy Nash equilibrium in $[A_i, \Pi_i: 1 \ldots n]$. For all $i = 1 \ldots n\, P_i = \theta$.

Lemma 2 shows that the only possible pure-strategy Nash equilibrium involves all firms setting the competitive price. Under Assumption 3 this implies outputs $X_i = S(\theta)$. As such, Lemma 2 mimics Shubik's result that in Bertrand–Edgeworth models the only possible equilibrium is competitive (Shubik, 1959).

Theorem 1 gives a sufficient condition for the competitive outcome to be an equilibrium. We have been unable to find any enlightening necessary condition beyond the equilibrium definition itself.

THEOREM 1. *Consider* $[A_i, \Pi_i: i = 1 \ldots n]$. *A pure-strategy Nash equilibrium exists if* $\sigma(\theta) \geqslant S(\theta) . n/(n-1)$.

Proof. We will prove Theorem 1 by construction. From Lemmas 1–2 we know that in any equilibrium $(\mathbf{P}, \mathbf{q})$ that exists, for all firms i:

$$P_i = \theta,$$
$$d_i(\boldsymbol{\theta}, \mathbf{q}) = S(\theta),$$
$$\Pi_i(\boldsymbol{\theta}, \mathbf{q}) = \zeta(\theta).$$

We will now show that if firms choose $q_i = \sigma(\theta)$ and the condition of Theorem 1 is met, $(\boldsymbol{\theta}, \boldsymbol{\sigma}(\theta))$ is indeed an equilibrium. First, note that no firm will wish to undercut: from Assumption 3 and definition 1, its profits would then fall to $\zeta(\theta-\epsilon)$ for $\epsilon \geqslant 0$. Will any firm wish to raise its price as in the standard Bertrand–Edgeworth model – see Dixon (1987 theorem 1)? Not if Theorem 1's condition is met, since the $(n-1)$ firms still setting θ will be willing to raise output to $(n-1)\,\sigma(\theta)$, hence meeting all demand $b(\theta) . Y(H)$.

Formally, for $\epsilon > 0$ $d_i[\theta+\epsilon, \theta_{-1}, \boldsymbol{\sigma}(\theta)] = 0$

if $$(n-1) . \sigma(\theta) \geqslant b(\theta) . Y(H) = n . S(\theta)$$

so $$\sigma(\theta) \geqslant S(\theta) . n/(n-1).$$ QED.

There will in general be many equilibria, so long as firms are willing to offer to supply enough output at θ to make it not worth while for a Nash-deviant to raise its price. In particular, the condition given by the Theorem is sufficient for n-1 firms to be able to satisfy all the demand at θ: clearly, they may leave unsatisfied demand whilst still making a price rise unprofitable. Furthermore, note that in the absence of the no-bankruptcy constraint an equilibrium would always exist for $n > 1$: each firm could for example offer to serve the whole market at the competitive price.[3] The Theorem gives a condition for existence that does not violate no-bankruptcy.

How robust is the equilibrium? Let us first consider the standard Bertrand equilibrium where firms have the same constant average/marginal costs and set price equal to this unit cost: this equilibrium involves firms playing a weakly dominated strategy, and is not trembling-hand perfect (see Harrington (1990) for a more formal statement and analysis). Clearly, setting price equal to unit cost yields zero profits whatever other firms do, whilst higher prices sometimes yield positive profits. Also, if there is any chance that the other firm(s) will set a price above unit cost, it will pay a firm to set its own price above unit cost. The equilibrium proposed in this paper is no better than in the standard Bertrand equilibrium. As should be clear from the proof of Lemma 1, the equilibrium strategy $(\theta, \sigma(\theta))$ is weakly dominated by $(\theta, S(\theta))$. Furthermore, if firms 'tremble', or there is any uncertainty introduced into demand, the firms will not be willing to offer to trade in excess of $S(\theta)$. However, in defence of the proposed equilibrium, we would argue that it can be used for a much more general class of cost functions than the standard Bertrand and Bertrand–Edgeworth approaches.

An example: Duopoly with Quadratic Costs

Let us take the classic case of duopoly $n = 2$ with quadratic costs: $c(x) = c.x^2$. Hence $S(P) = P/2c, \sigma(P) = P/c$. The exact nature of demand is not important for establishing existence: simply let the competitive price be θ. We propose that $(\boldsymbol{\theta}, \boldsymbol{\sigma}(\theta))$ is an equilibrium: each firm sets $P_i = \theta$, produces $X_i = \theta/2c$, and offers to sell $q_i = \theta/c = 2.X_i$. Note that the condition of Theorem 1 is met.

To check that this is indeed an equilibrium, note that if either duopolist raises its prices above θ, the other will meet all demand at θ (since $q_i = 2.S(\theta)$), and there will be no demand for the higher-priced firm.

III. CONCLUSION

This paper has proposed a simple solution to the non-existence of pure-strategy equilibria in Bertrand–Edgeworth models. By generalising Edgeworth's trading process to allow firms to choose how much they are willing to sell as well as their price, we establish both that pure-strategy equilibria exist under fairly weak conditions, and that any equilibria yield the competitive outcome.

This result is of interest for two reasons. It provides a simple non-cooperative foundation for the competitive outcome, and shows how price-setting firms might behave as if they were price-takers. Unlike the Cournot paradigm, it does not rely on large numbers; unlike the standard Bertrand approach it does not rely on constant marginal/average costs. The price–quantity game presented can be applied to model competitive equilibrium in any market with convex costs and price-setting firms. Secondly, the model solves the existence

[3] Allen and Hellwig (1986) make this point.

problem endemic in Bertrand–Edgeworth models and does so without generating non-competitive equilibria.

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Date of receipt of final typescript: August 1991

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APPENDIX

Proof of Lemma 2

We will prove Lemma 2 in two stages. First, that in any equilibrium firms must set the same price: secondly, that the competitive price is the only single-price equilibrium (SPE).

To show that no multiple-price equilibria exist, assume the contrary to derive a contradiction, so that in equilibrium for some i,j (to avoid trivialities) $\bar{P} > P_i > P_j \geqslant \theta$. Either $d_i = 0$, or $d_i > 0$. If $d_i = 0$, then $\Pi_i = 0$. The firm can always increase its profits to $\zeta(\theta) > 0$ by setting $P_i = \theta$. From Definitions 1, 2, when a firm sets $P_i = \theta$ its demand is no less than θ, given that from Lemma 1 no lower-priced firm will choose $q_k > S(P_k)$, the desired contradiction. If $d_i > 0$, then there is excess demand for the lower-priced firms, so that one can raise its price slightly and sell the same amount, thus raising profits, the desired contradiction. Hence we have a contradiction both when $d_i = 0$ and $d_i > 0$, thus ruling out multiple-price equilibria.

To show that θ is the only possible SPE, let us assume the contrary, $P \neq \theta$. Either $P > \theta$ or $P < \theta$. If $P < \theta$, there will be excess demand (from Lemma 1 no firm will choose $q_i > S(P)$:

$$d_i(P, \mathbf{q}) \geqslant b(P)\,[Y(H) - (n-1)\,S(P)\,Y^*(P)] > S(P).$$

Since b is continuous, by raising its prices to some $P+\epsilon$, i can still sell $S(P)$, thus raising profits by $\epsilon S(P)$. If $P > \theta$, there will be excess supply.

$$d_i(P, \mathbf{q}) = \frac{b(P)}{n}\,Y(H) < S(P).$$

Hence $\Pi_i(P, \mathbf{q}) < \zeta(P)$. By undercutting and choosing $q^1 = S(P-\epsilon)$:

$$\sup_{\epsilon > 0} \Pi_i[P-\epsilon, P, S(P-\epsilon), q_{-1}] = \zeta(P).$$

Hence the firm can increase its profits, the desired contradiction. Since we have contradiction by assuming an SPE $P > \theta$ and $P < \theta$, it follows that if an SPE exists, $P = \theta$. QED.