Introductory Mathematics for Economics MSc's.
Lecture 2: Optimization and Specific Functions

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September 2009.

1. Quadratic Functions.

Quadratic functions take the form $y = c + bx + ax^2$. The linear function is a special case, where c=0.

These have a familiar shape. We can graph them easily. When x=0, we have y=c: so, a (the constant) is the intercept term. The derivatives are

$\frac{dy}{dx} = b + 2ax$ Hence, when x=0, b gives the slope (positive if b>0, negative if b<0, zero if b=0).

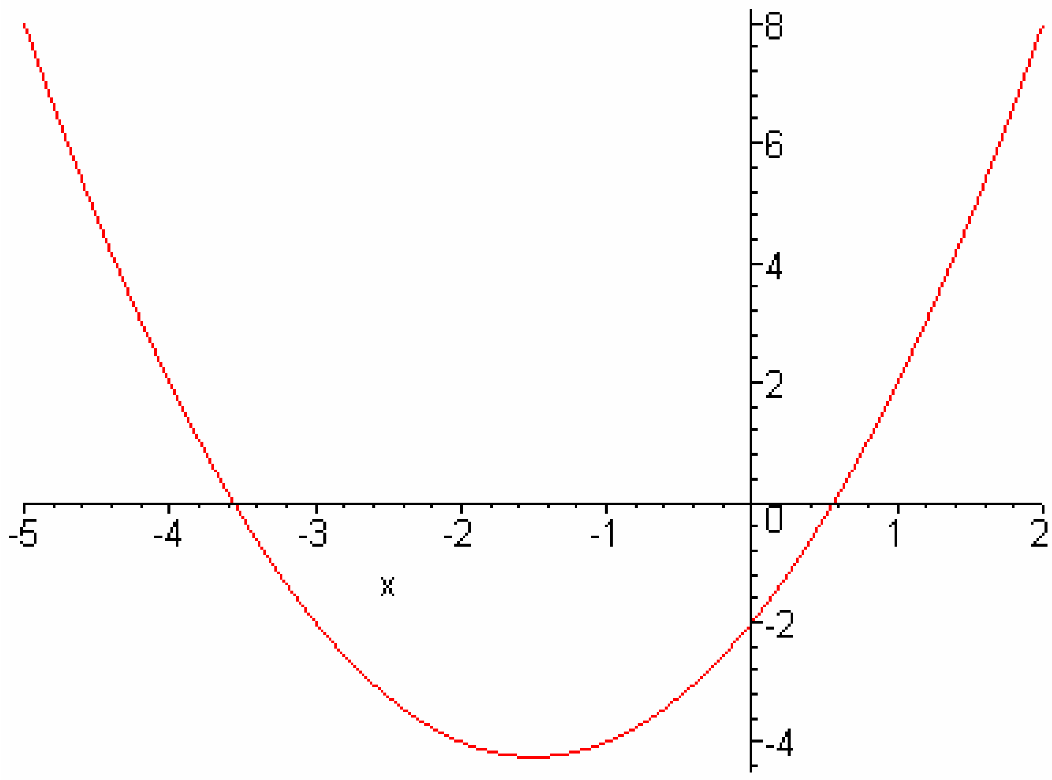
$\frac{d^2y}{dx^2} = 2a$ *a* determines whether the quadratic is concave (a<0; an upside down bowl) or *a>0*.convex (a bowl).

For example:

$$y = c + bx + ax^2 = -2 + 3x + x^2$$
$$a = 1; b = 3; c = -2$$

When x=0, y=-2, and the *slope* of the curve is 3 (upward sloping). Also, the quadratic is strictly convex since a>0.

Let us take a look:

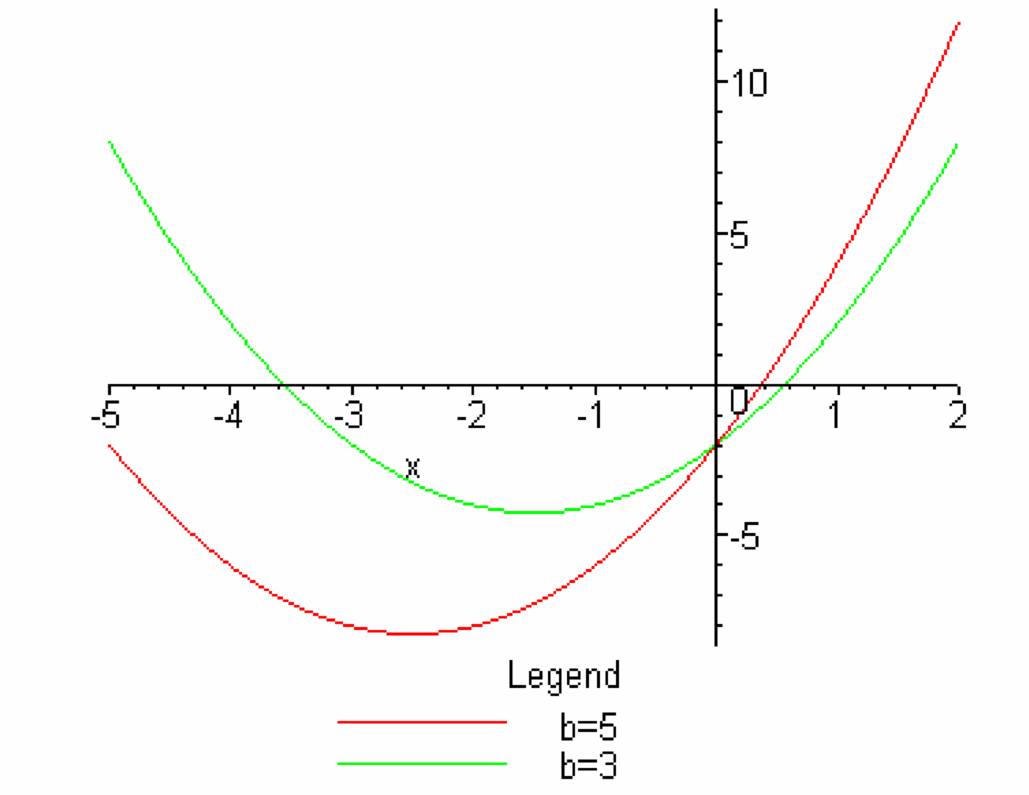


The general properties of the quadratic are captured by the three coefficients a,b,c.

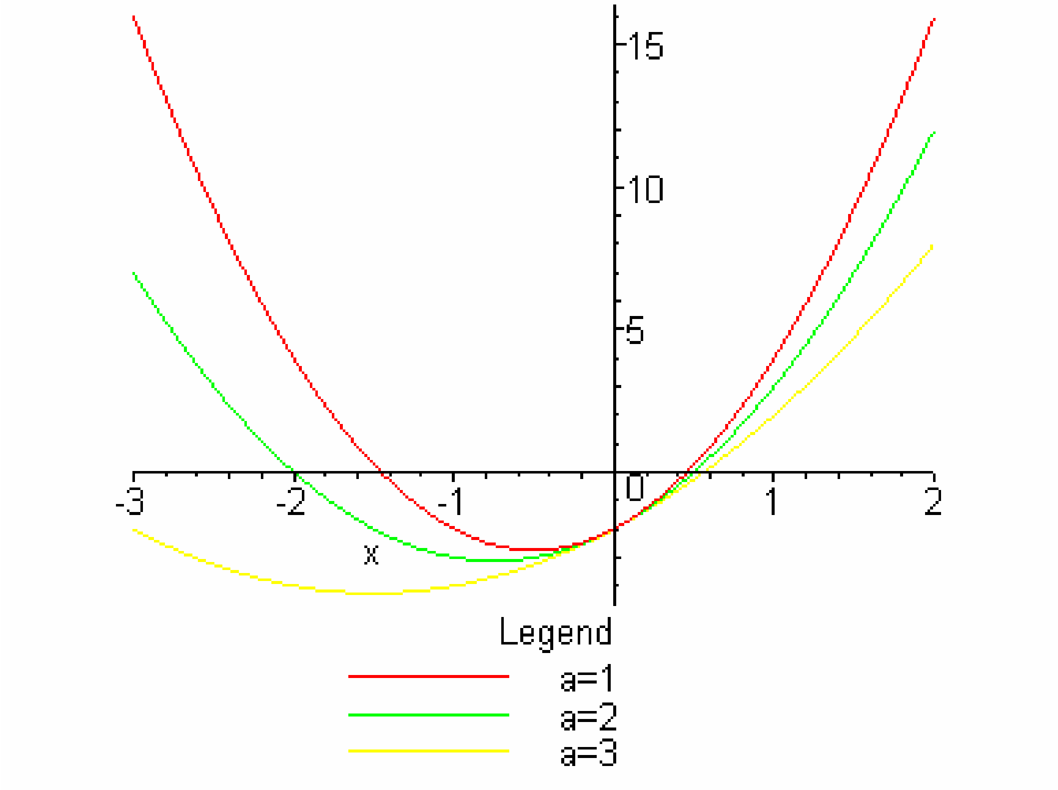
What happens if we vary them?

Varying c moves the quadratic up and down.

If we change b, the quadratic rotates around the intercept and compresses horizontally.



If we increase a, we make the compress the curve horizontally…..



Note: the legend is the wrong way around: red is $a=3$, and yellow $a=1$.

Now: we can ask the question what are the roots of the quadratic? When does it equal zero.

These are the values of x for which y=0

$$y = c + bx + ax^2 = 0$$

The solution to this is: $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

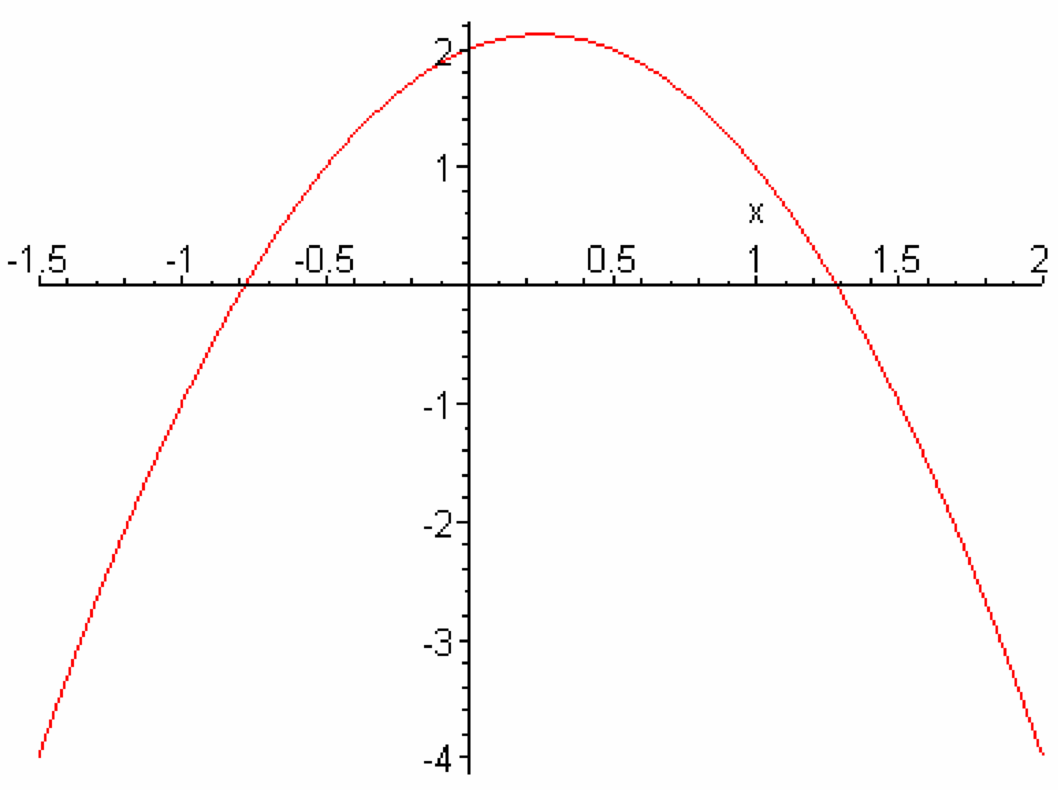
Let us take our example:

$$x = \frac{-3 \pm \sqrt{3^2 - 4(-2)}}{2} = -1.5 \pm \frac{\sqrt{17}}{2}$$

$$= \text{-3.5616, 0.5616} \qquad \text{(to 4 dp).}$$

This corresponds to the places where the quadratic intersects the horizontal x-axis (see above).

Let us take another example: a=-2; b=1; c=2: $y = 2 + x - 2x^2 = 0$ This is strictly concave:



When x=0, y=c=2.

When x=0, $\dfrac{dy}{dx} = b = 1$

What are the roots: from picture, about x=-0.79 and 1.3.

The formula?

$$x = \frac{-1 \pm \sqrt{1+16}}{-4} = \frac{1}{4} \pm \frac{\sqrt{1+16}}{4}$$

= -0.78078, 1.2808 (to 4 dp).

The formula predicts what we see!

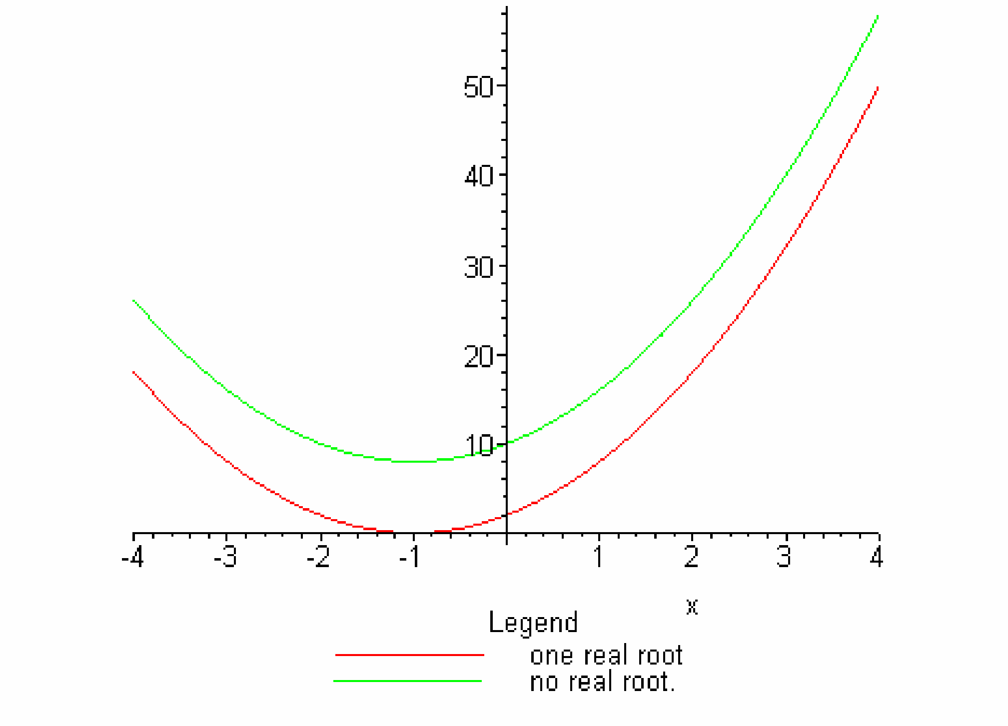
Now, three cases:

$b^2 - 4ac > 0$ As in the examples, two *real* roots.

$b^2 - 4ac = 0$ One real root $x = -\dfrac{b}{2a}$

$b^2 - 4ac < 0$ No real roots (imaginary roots…..)

Let us look at what the one real root and the no real roots cases look like:



What happens when there is no real root? We have imaginary roots: we take the square root of a negative number…..not needed for this course!

2: maximizing a Quadratic function.

Let us take the example of the concave quadratic function. What is the largest value of y and what is the value of x which yields this?

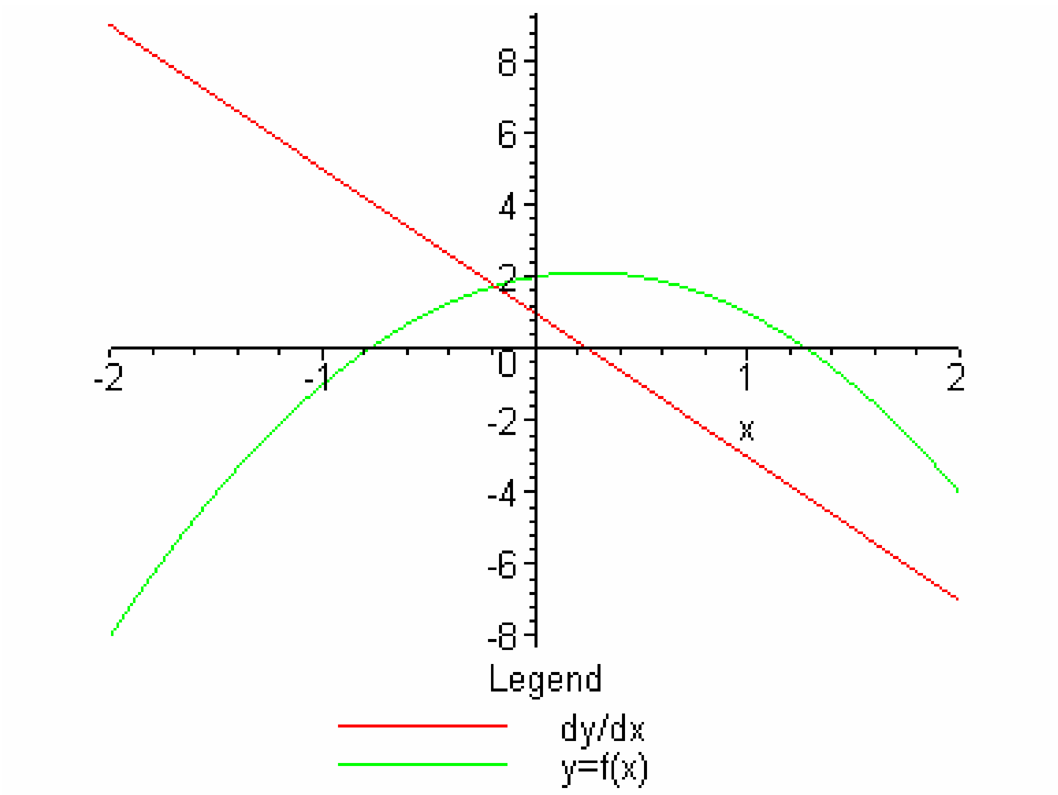
If we look at the quadratic, we can see it is like a hill. At the top, it is flat: the slope is zero.

If you are away from the top the slope is not zero: if you travel towards the top of the hill, the slope is positive (you are walking upwards); if you travel away from the top the slope is negative (you are walking downwards).

Let us plot the quadratic and its first derivative:

$$y = 2 + x - 2x^2$$

$$\frac{dy}{dx} = 1 - 4x$$



We can see that the derivative is decreasing in x.

When f(x) is increasing, the slope is strictly positive.

As x increases, the slope decreases.

At the top (maximum) of f(x), the slope is zero.

After the top, the slope is negative and gets more negative as x increases.

To solve for the maximum: solve $\dfrac{dy}{dx} = 1 - 4x = 0$, hence $\dfrac{dy}{dx} = 0 \Rightarrow x = 0.25$.

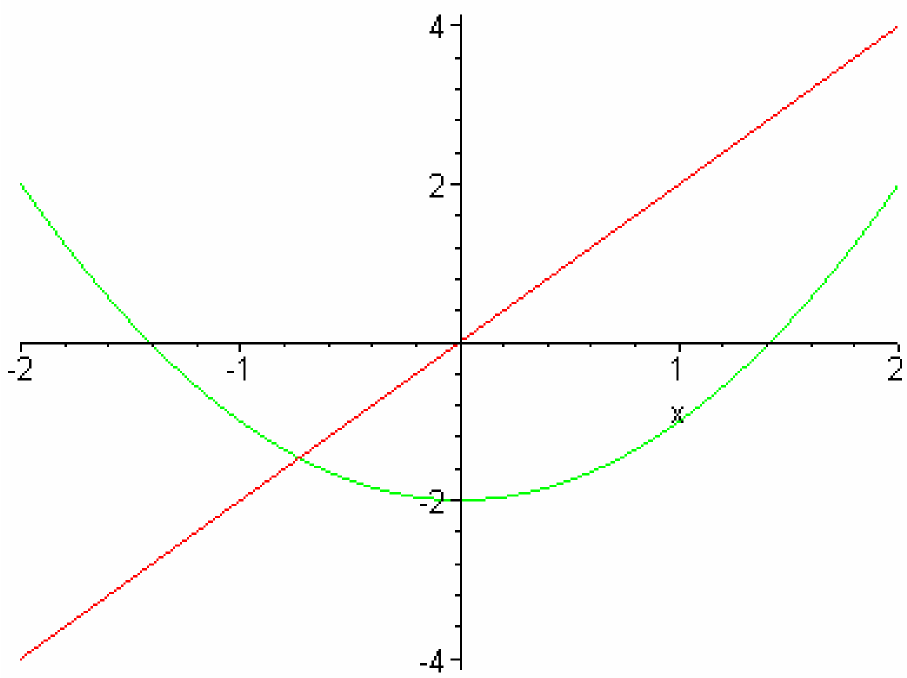
So, what is the maximum value of y? It is the value of the quadratic when x=0.25.

$$y = 2 + x - 2x^2$$

$$x = \frac{1}{4} \Rightarrow y = 2 + \frac{1}{4} - 2\frac{1}{16} = \frac{9}{8} = 2.125$$

So: maximization made simple!

Rule 1: Check if the function is strictly concave for all values of x. If it is, then differentiate the function and solve for the value of x which makes the derivative zero.



What happens if the function is strictly convex for all values of x? Then finding the value of x for which the derivative is zero is finding the bottom of a hole, or minimizing the function. As you move towards the bottom you move downwards, and as you reach the bottom you flatten out; as you leave the bottom you move back up. Here we have the quadratic $y = x^2 - 2$

The minimum $\frac{dy}{dx} = 2x = 0 \Rightarrow x = 0$

$$y = -2.$$

3: Applications of Quadratics.

In economics, we use quadratic functions a lot.

Profit Function. Profits π equal revenue minus costs.

$P(x) = A - B.x$ Linear demand

$C(x) = \frac{x^2}{2}$ Strictly convex costs.

Profits are then a *Concave* quadratic function of output: Linear demand implies revenue is a *concave* function of output (this is the square bracket).

$$\pi = x.P(x) - C(x)$$
$$= x(A - Bx) - \frac{x^2}{2} = \left[Ax - Bx^2\right] - \frac{x^2}{2}$$
$$= A.x - \left(B + \frac{1}{2}\right)x^2$$

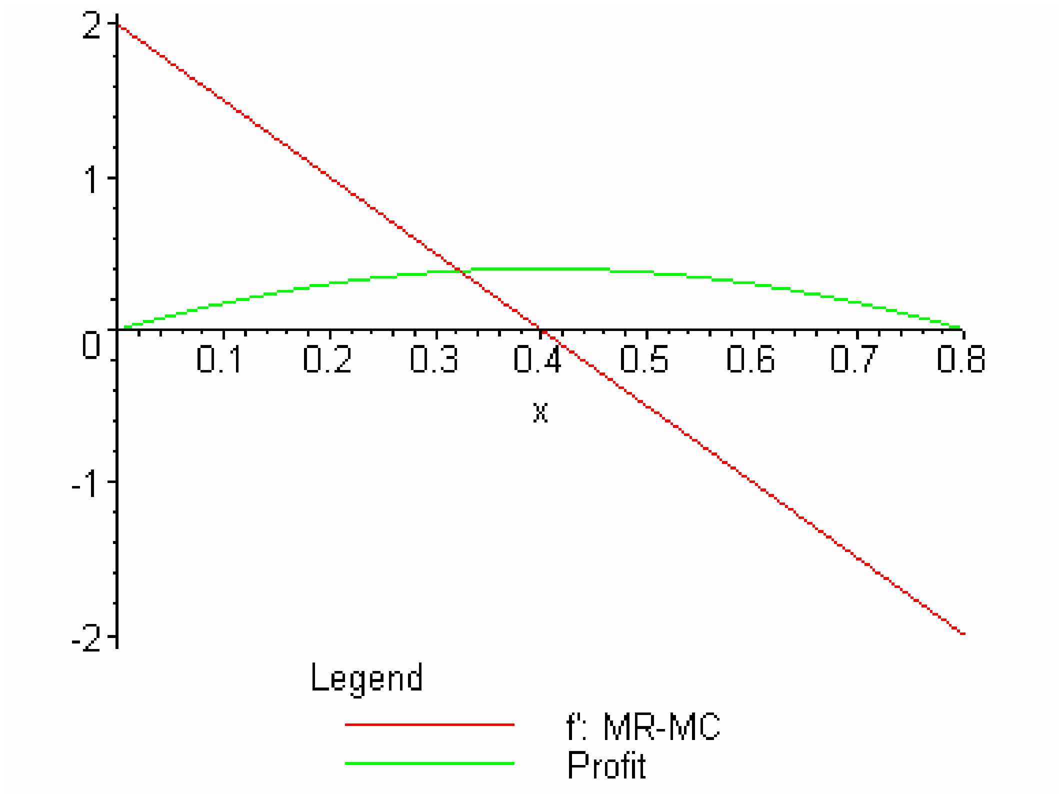
With linear demand and quadratic costs, profits become a concave quadratic.

What is the output that maximizes profits?

Example: A=2, B=2

$$\pi = 2x - \frac{5}{2}x^2; \quad \frac{d\pi}{dx} = 2 - 5x = 0 \Rightarrow x^* = \frac{2}{5} = 0.4$$

$$\pi^* = 2.\frac{2}{5} - \frac{5}{2}\frac{4}{25} = \frac{4}{5} - \frac{2}{5} = \frac{2}{5} = 0.4.$$



The condition $\frac{d\pi}{dx} = 0$ is equivalent to the condition that MR-MC=0.

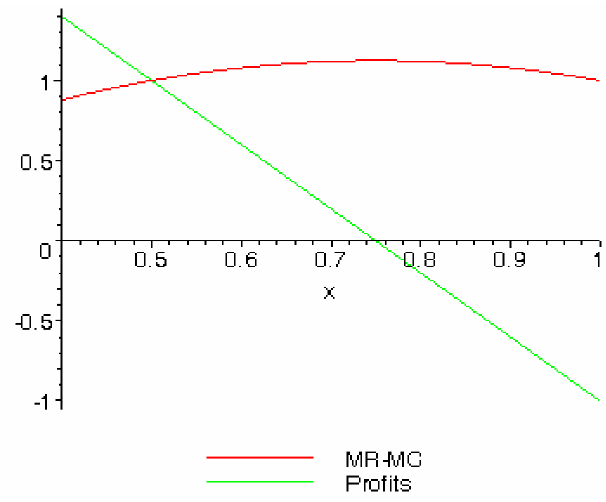
Another example: *Linear costs.*

Suppose that the cost function is $C(x) = c.x$. Then we have

$$\pi = x.P(x) - C(x)$$
$$= x(A - Bx) - cx = (A - c)x - Bx^2$$

Hence if c=1 A=4, B=2:

$$\pi = 3x - 2x^2; \quad \frac{d\pi}{dx} = 3 - 4x = 0 \Rightarrow x^* = \frac{3}{4} = 0.75$$
$$\pi^* = 3\frac{3}{4} - 2\frac{9}{16} = \frac{9}{4} - \frac{9}{8} = \frac{9}{8} = 1.125$$



Maximization.

In economics, we often model behaviour as optimization: agents try to maximize something: profits, utility and so on.

This can either be *constrained* maximization or *unconstrained* maximization. We deal here with unconstrained maximization. We choose x to maximize a function $y = f(x)$, possibly over some restricted range of x. For example, in economics many variables must be positive (output, price and so on).

We write this as:

$$y^* = \max \ f(x)$$
$$x^* = \arg\max \ f(x)$$

For example, a firm's profit can be seen as a function of output: the output that maximises profit is called the firms supply, the resulting profits are the maximum profits.

A Necessary condition for a maximum is that the first order derivative is zero

$$\frac{dy}{dx} = 0 \text{ at } x^*$$

This is called the *first order condition* for a maximum (first order means the first order derivative).

We also have second order conditions, relating to the second order derivatives.

Second order condition:

S01. The function is globally concave: $\frac{d^2y}{dx} < 0$ for all x

SO2. The function is locally concave at x*: $\frac{d^2y}{dx} < 0$ at x*

The first order condition and S01 are sufficient for a *global* maximum.

The first order condition and S02 are sufficient for a *local* maximum.

For a *minimum* we have same conditions, but convexity replaces concavity.

Examples (Maths):

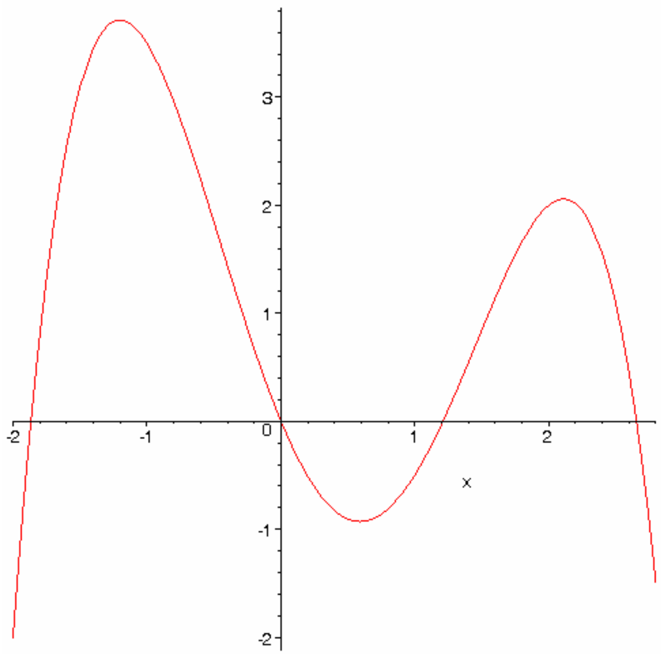
We have already seen case of a concave quadratic: this is globally concave. Hence we have a maximum whenever the first order condition is satisfied.

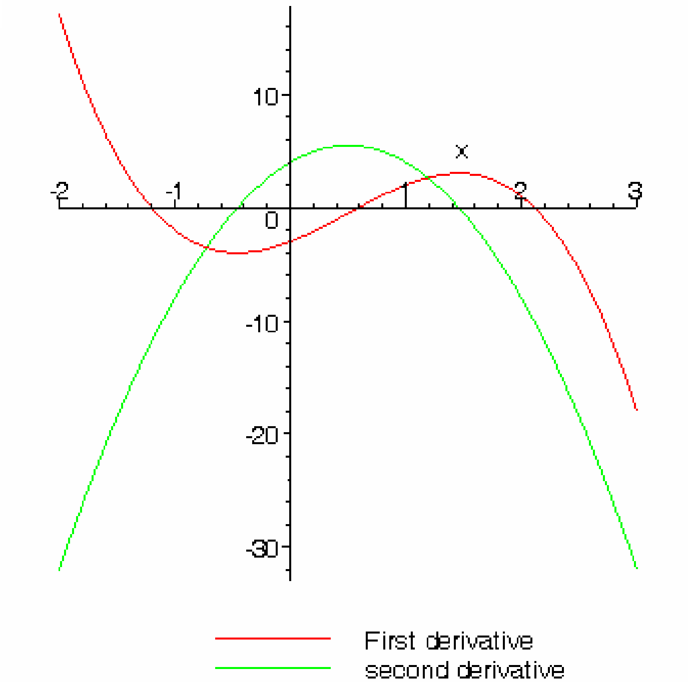
Now, consider the function $y = -3x + 2x^2 + x^3 - \frac{1}{2}x^4$

If we differentiate this, we get:

$$\frac{dy}{dx} = -3 + 4x + 3x^2 - 2x^3$$

$$\frac{d^2y}{dx^2} = 4 + 6x - 6x^2$$





Now, we can see that from the green line (second derivative), the function is convex for a range of values of x (roughly -0.4 to 1.4), but concave everywhere else.

We can see that the first order derivative is equal to zero three times: -1.20, 0.59, 2.11

The first of these is the *global* maximum: the first derivative is zero, and the function is concave when $x = -1.2$ (the green line is negative).

The second is a *minimum*: when x=0.59, the second derivative (green line) is positive.

The third is a *local* maximum.

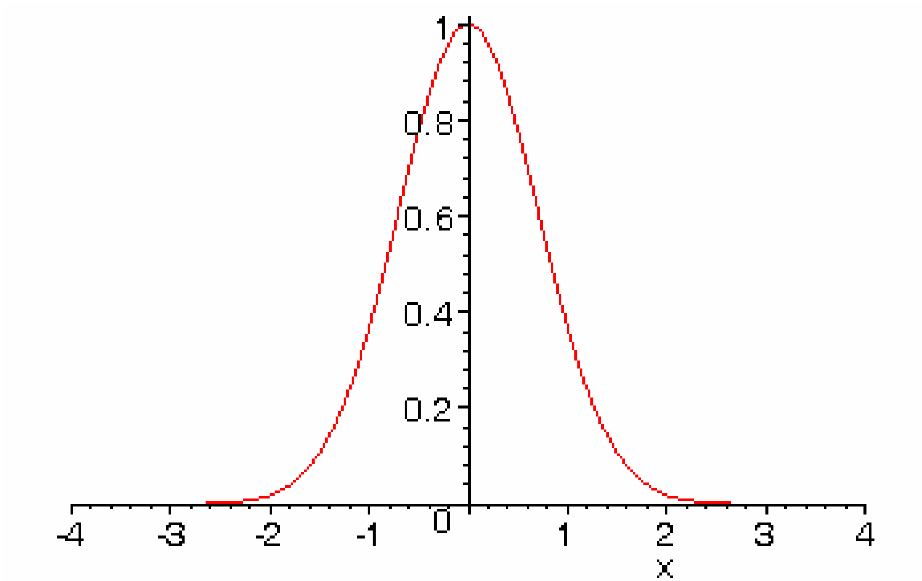
Example 2.

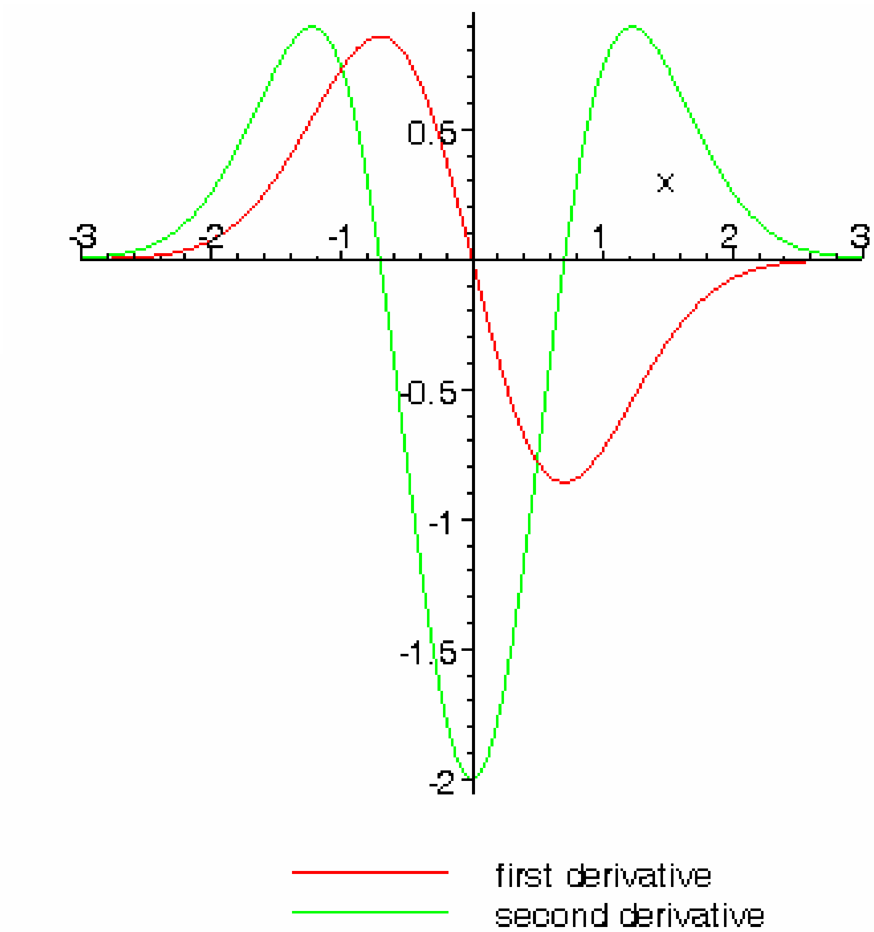
The normal distribution:

$$y = e^{-x^2}$$
$$\frac{dy}{dx} = -2xe^{-x^2}$$
$$\frac{d^2y}{dx^2} = e^{-x^2}(4x^2 - 2)$$

.

Note that this function is not globally concave: it is convex away from the "mean" x=0, and concave around the mean.





Note that this function is not globally concave: it is convex away from the "mean" x=0, and concave around the mean. There is only one value of x=0 where the first derivative is zero: that happens when x=0 (the mean). Here it is concave, so it is a local maximum: since there is no other local maximum, it is also the global mean.

Example from Economics.

In Economics, we almost always assume that the function to be optimized is always strictly concave. That means that the first order conditions give us a unique solution, which is the global maximum.

Example 1: Profit maximization by the monopolist. We already saw that with linear demand and either quadratic costs or linear costs, the profit function is quadratic. Let us take another case or constant elasticity of demand.

$P = x^{-\frac{1}{\varepsilon}} \quad C(x) = cx$ Constant elasticity demand: linear costs

$\Pi = x^{1-\frac{1}{\varepsilon}} - cx$ Profits as a function of output.

$\frac{d\Pi}{dx} = \left(1 - \frac{1}{\varepsilon}\right) x^{-\frac{1}{\varepsilon}} - c$ The first derivative

$\frac{d^2\Pi}{dx^2} = -\frac{1}{\varepsilon}\left(1 - \frac{1}{\varepsilon}\right) x^{-\frac{1}{\varepsilon}-1} < 0$ The second derivative: always strictly negative (at least for x>0) when ε>1 (this is assumed).

Profit maximization: First order condition

$$\frac{d\Pi}{dx} = \left(1 - \frac{1}{\varepsilon}\right) x^{-\frac{1}{\varepsilon}} - c = 0 \Rightarrow P\left(1 - \frac{1}{\varepsilon}\right) = c$$

Marginal revenue equals marginal cost!

$$P = \frac{\varepsilon}{\varepsilon - 1} c$$

Thus, price is a “markup” over marginal cost (ε>1 implies P>c)

$$x = \left[\frac{\varepsilon}{\varepsilon - 1} c\right]^{-\varepsilon}$$

Example 2: Profit maximization by a competitive firm.

Competitive firms behave as price takers: they choose output to maximize profits given the price.

$\Pi = Px - c(x)$ Profits.

$\frac{d\Pi}{dx} = P - c'(x)$ First order condition $P = MC$

$\frac{d^2\Pi}{dx^2} = -c''(x)$ Second order: so long as costs are *convex*, profits are globally *concave* in x.

Explicit Function:

$$c(x) = 2x^2$$

$$MC = 4x; \qquad \frac{d\Pi}{dx} = 0 \Rightarrow P = 4x \Rightarrow x = \frac{P}{4} = s(P) \quad \text{Supply function}$$

$$c''(x) = 4 > 0 \Rightarrow \frac{d^2\Pi}{dx^2} = -4 < 0$$

Exponential Function. $e = 2.7182818283....$

e^x can be defined as an infinite series.

$$e^x = 1 + x + \frac{x^2}{2!} + ...\frac{x^n}{n!}....$$

Where n! is called n factorial: n!=n(n-1)(n-2)(n-3)…3.2.

Thus:

$$0! = 1$$
$$1! = 1$$
$$2! = 2x1 = 2$$
$$3! = 3x2 = 6$$
$$4! = 4x3x2 = 24$$

Now we can see why $\frac{de^x}{dx} = e^x$.

$$e^x = 1 + x + \frac{x^2}{2!} + \ldots \frac{x^n}{n!} \ldots.$$

$$\frac{de^x}{dx} = 0 + 1 + x + .. \frac{x^{n-1}}{(n-1)!} \ldots. = 0 + e^x$$

If you differentiate e^x, get an infinite series of terms which are the same as you started with (each one moves one to the right with a zero at the start).

The uses of e in economics: continuous time.

1. Growth. If a variable has an initial value of x(0) at time t=0, and grows at a rate g, at time t it is $x(t) = x(0).e^{gt}$
2.Compound interest. If you invest x(0) at time t=0, and the rate of interest is r, at time t it is worth $x(t) = x(0).e^{rt}$

3. **Discounting**. If you are paid x(t) at period t, what is it worth now (t=0)?

How much would you have to invest now in order to get x(t) at time t? From 2

$$x(0) = \frac{x(t)}{e^{rt}} = x(t)e^{-rt}$$

Discrete time: compound interest.

Financial contracts are usually defined for a given period: a bond is for a month or a year.

A given principal (sum of money) P is invested for 1 year at annual interest rate *i*, the value at the end of the year is:

$$S = (1+i)P$$

Now, suppose that the sum is invested for two periods of 6 months at an semi-annual interest rate i^6. The return is then:

$$S = (1 + i^6)^2 P$$

Now, what is the value of the semi-annual interest rate that gives the same return as the annual interest rate i?

$$(1+i) = (1+i^6)^2 = 2i^6 + (i^6)^2 + 1$$

$$i = 2i^6 + (i^6)^2 \Rightarrow 2i^6 + (i^6)^2 - i = 0$$

Using the quadratic formula to solve for i^6 we have *a=1, b=2, c=-1.*

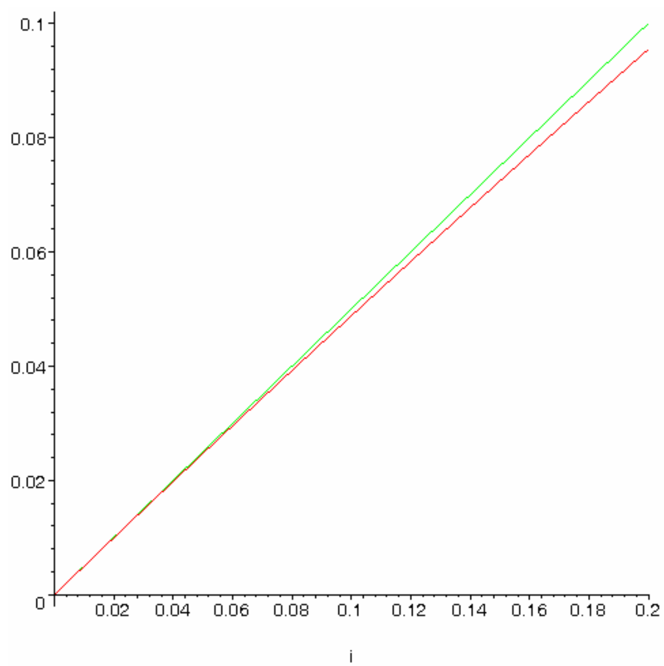
$$i^6 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{4+4i}}{2} = -1 + \frac{\sqrt{4(1+i)}}{2} = -1 + \sqrt{(1+i)}$$

Since in economics i is a small number (e.g. 0.05, 0.1)) $i^6 \approx \frac{i}{2}$.

That is, the semi-annual rate is roughly half the annual rate.

For small i^6, $(i^6)^2 \approx 0$. (e.g. i^6=0.05 means $(i^6)^2$=0.0025).

We can see this approximation is quite good:



However, since there is compounding, the semi-annual rate is a little less than half the annual rate.

Compounding: first six months get $S' = (1+i^6)P$. In the second sixth months, you get interest on *P* plus *interest on the interest earned* in the first six months $S = (1+i^6)P + (1+i^6)i^6P = (1+i^6)^2 P$.

Now, suppose you take the annual interest rate *i*, and divide it up into *n* sub-periods, then

$$S = (1+\frac{i}{n})^{nt}$$

What if we compound the interest continuosly, $n \to \infty$. First, note that an alternative (and equivalent) definition of *e* is:

$$e = \underset{n\to\infty}{Lim}(1+\frac{1}{n})^n$$

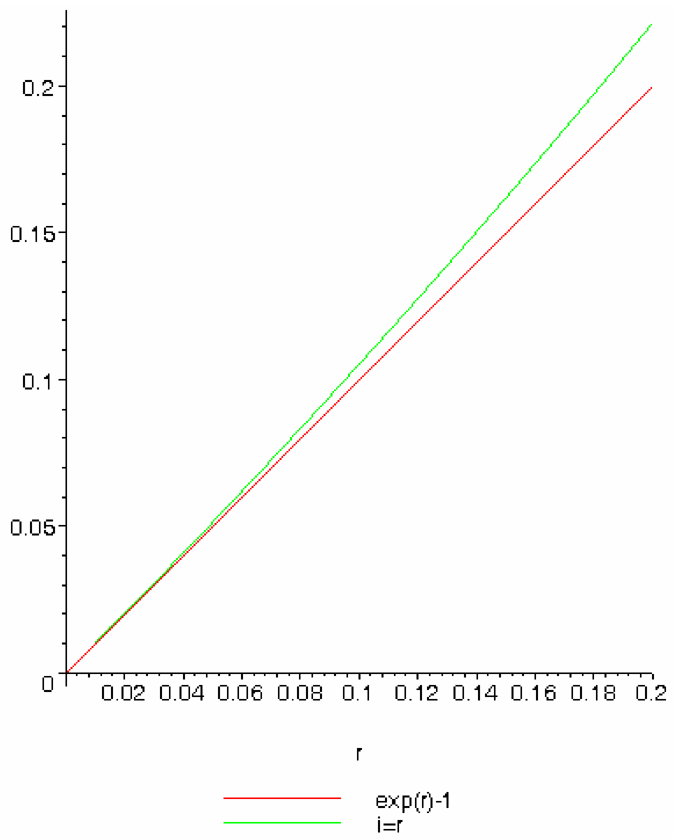
With instantaneous interest rate *r* we have

$$\underset{n\to\infty}{Lim}(1+\frac{r}{n})^{nt} = e^{-rt}$$

So, an annual interest rate of *i* is equal to a continuous compounding rate of *r* where

$$(1+i) = e^r \Rightarrow i = e^r - 1$$

Note again, since r will be small in economic applications, $e^r \approx 1+r$ (remember the infinite series: all the higher order powers of *r* become negligible). Hence for small *i* and *r* $i \approx r$.



The annualised interest rate *i* is on the vertical axis: the constant compounding rate *r* is on the horizontal axis. For small interest rates, the two are almost the same. However, whilst compounding may not appear to make much difference with small interest rates, if a lot of money is involved (*P* is $billions!), it pays to be exact……

Logarithms.

We only deal with natural logarithms. The log of a number is the power to which *e* must be raised to equal that number:

$$x = e^{\ln x}$$

Thus taking the log of a number x is the inverse of raising e to the power x.

Rules of logs:

$$\ln(xy) = \ln x + \ln y$$

$$\ln x^a = a \ln x$$

If you have multiplicative expressions or powers, taking logs make them linear in logs, or log-linear.

For example: Cobb-Douglas

$$Y = K^{\alpha} L^{1-\alpha}$$

$$\ln Y = \alpha \ln K + (1-\alpha) \ln L$$

Note: if x is small (close to zero), then $\ln(1+x) \approx x$.

Taylor Approximation.

We can approximate any function: an approximation can be good or bad!

Economists often use

Linear Approximation. This is good if we are approximating something close to a point.

We take a point x_0. The real function is $y = f(x)$. We approximate this by:

$$y \approx f(x_0) + (x - x_0).f'(x_0).$$

Since by definition we know $y_0 = f(x_0)$, this can also be written as

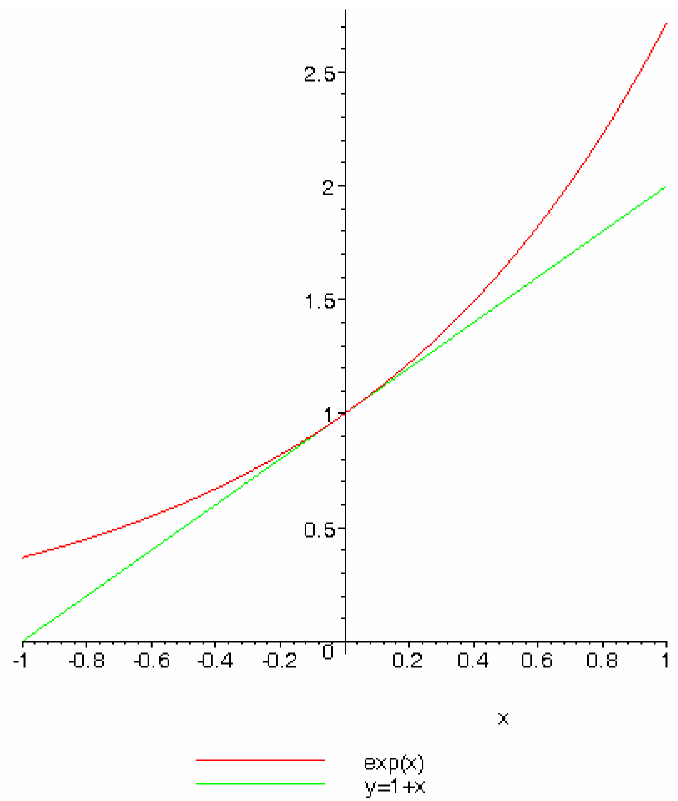
$$y - y_0 \approx (x - x_0).f'(x_0).$$

For example: $y = e^x$. Let us approximate it around *x=0*:

$\begin{array}{l} y_0 = 1. \\ f' = 1 \end{array}$ Hence $y \approx 1 + x$

Is this good? Let's take a look:

It gets worse the further away you go from *(1,1)*.



Second-order Approximation.

Taylor's Theorem. You can approximate any function by an nth order polynomial (i.e. has x^n in it). In economics we often use *second order polynomials*: i.e. Quadratic.

$$y \approx y_0 + f'(x_0)(x - x_0) + \frac{f''}{2}(x - x_0)^2$$

Note: both the derivatives f', f'' are evaluated at x_0. The second derivative is divided by two.

How does this do? Well, at $(y = 0, x = 0)$, $f'' = e^0 = 1$. Hence, $y \approx 1 + x + \frac{x^2}{2}$

Let us see how this does!

The second-order is much nicer, but eventually not so good!

Brings us back to quadratics.

THE END.