INTRODUCTORY MATHEMATICS FOR ECONOMICS MSC'S.
LECTURE 4: LINEAR ALGEBRA AND SIMULTANEOUS EQUATIONS.

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SEPTEMBER 2009.

Simultaneous Equations.

Consider two equations in two unkowns (x,y):

$$ax + by + c = 0$$
$$\alpha x + \beta y + \gamma = 0$$

Where a,b,c,α,β,γ are constants. There are a few questions we can ask about this.

1. Does a solution exist?
2. If a solution exists, is it unique?
3. If a solution exists, how do you solve for it.

Let us first ignore the first two questions: let us just go ahead and try to solve for x and y "manually". To do this, we can first express x as a function of y in the second equation, and then substitute into the first to solve for y.

$$x = -\left(\frac{\beta}{\alpha}y + \frac{\gamma}{\alpha}\right)$$

$$-a\left(\frac{\beta}{\alpha}y + \frac{\gamma}{\alpha}\right) + by + c = 0$$

$$y\left[b - \frac{a\beta}{\alpha}\right] + c - \frac{a\gamma}{\alpha} = 0$$

$$y = -\left[\frac{c\alpha - a\gamma}{b\alpha - a\beta}\right]$$

Now, note that *this solution is only valid if* $b\alpha - a\beta \neq 0$. When we divide both sides of line 3 by $b\alpha - a\beta$, this is only valid if it is non-zero (you cannot divide by 0 in algebra except in special circumstances).

However, if $b\alpha - a\beta \neq 0$, then there exists a unique solution to the problem. Given y, we can go back to line 1 to recover x.

Now, this term $b\alpha - a\beta$ is called the *Determinant* of the system, often denoted Δ.

Essentially we are looking at two straight lines: do they cross (the solution is a point where they cross).

If the determinant is non-zero, then it means that they must cross: they have different slopes.

To see this, we can rewrite the two equations to express x as a function of y:

$$x = -\frac{b}{a}y - \frac{c}{a}$$

$$x = -\frac{\beta}{\alpha}y - \frac{\gamma}{\alpha}$$

Now, we can see that if the determinant is zero, this is the same thing as saying that the slopes of the two lines are equal, i.e. they are parallel

$$\Delta = b\alpha - a\beta = 0 \Rightarrow b\alpha = \beta a \Rightarrow \frac{b}{a} = \frac{\beta}{\alpha}$$

Or course we know that two parallel lines never meet! Well, almost: if they are the same line, then they always meet (there are an infinity of solutions). This happens when $c\alpha - \gamma a = 0$.

Hence we have answered our 3 questions.

1. When will a solution exist? When the Determinant is non-zero $\Delta = b\alpha - a\beta \neq 0$, there exists a unique solution.

2. If the Determinant is zero, then there are two cases:
   (a) if $c\alpha - \gamma a \neq 0$, then the lines are parallel and have different intercepts in (x,y), so no solution exists.
   (b) if $c\alpha - \gamma a = 0$, the two lines are the same, and an infinite number of solutions exist.

In the books, you will often see the term "linear independence" associated with a non-zero determinant, and the phrase "full rank". These refer to the notion that if in some sense, two parallel lines represent the same relationship, just re-scaled. We say that when you have two parallel lines, they are a linear transformation of each other: the coefficients on x and y in one equation are the same as in the other except for a multiplicative constant. A non-zero determinant is also called "non-singular" and a zero determinant "singular".

Using Determinants to solve linear systems.

Cramer's Rule is a simple way to solve a linear system by calculating two Determinants and taking their ratio.

Gabriel Cramer (1704-1752)

Swiss Mathematician, developed Cramer's Rule (published (1750).

Let us take the two equation system and write it in the following form:

$$\begin{aligned} ax + by &= c \\ \alpha x + \beta y &= \gamma \end{aligned}$$

(note, this is not exactly the same: the constant have the "wrong" sign: but since we did not specify whether they were negative of positive, this does not matter!)

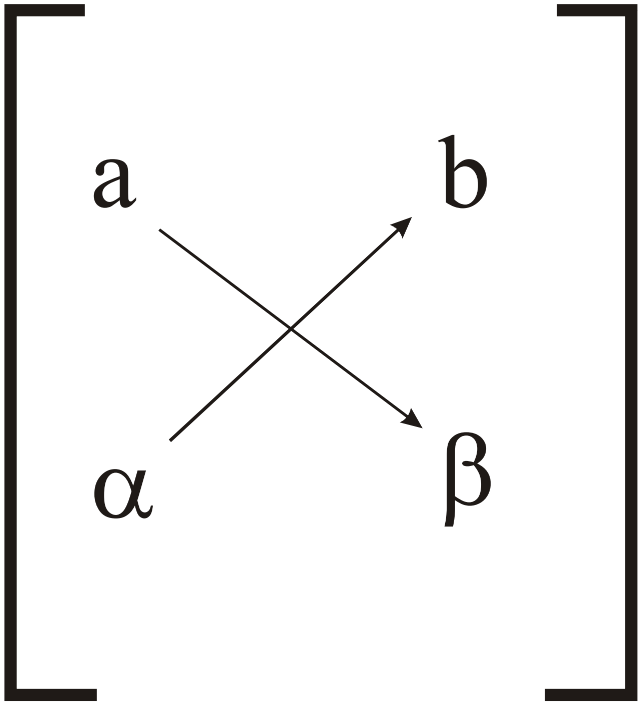
You can use Matrix notation for this: the system can be written:

$$\begin{bmatrix} a & b \\ \alpha & \beta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c \\ d \end{bmatrix}$$

The determinant of the 2x2 system (two linear equations, two variables) is defined by the coefficients on the variables x and y:

$$\Delta = \beta a - \alpha b.$$

(NB. Note: in a two by two system, the determinant will have a different sign depending on which equation you put first. This does not matter: when you calculate things, you just need to keep the order of the equations the same and you will get the same answer).



The rule for calculating the determinant of a 2x2 matrix.

Mutiply the leading diagonal (a times β) and subtract the product of the off diagonal (minus α times b).

The Determinant of a Matrix is also written as the matrix with straight lines at the side:

$$\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{21}a_{12}$$

Cramer's rule.

Remember: $\begin{bmatrix} a & b \\ \alpha & \beta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c \\ d \end{bmatrix}$

Now: the variable x is multiplied by the first column (a and α), the variable y by the second column in the matrix (b and β).

Cramer tells us to define two new determinants starting from the original 2x2 matrix of coefficients.

First, we replace the first column (the one that multiplies x) by the constants and take the determinant.

$$\Delta_x = \begin{vmatrix} c & b \\ \gamma & \beta \end{vmatrix} = c\beta - \gamma b$$

Second, we take the second column, and replace it with the constants: $\Delta_y = \begin{vmatrix} a & c \\ \alpha & \gamma \end{vmatrix} = a\gamma - \alpha c$

Cramer's Rule:

The solution (x,y) can be written as the ratio of determinants:

$$x = \frac{\Delta_x}{\Delta} = \frac{c\beta - \gamma b}{\beta a - \alpha b}$$

$$y = \frac{\Delta_y}{\Delta} = \frac{a\gamma - \alpha c}{\beta a - \alpha b}$$

Note: this rule only works for non-zero Δ.

Let's try it!

$$\begin{aligned} 3x + 4y &= -6 \\ -7x + 5y &= 2 \end{aligned} \qquad \Delta = \begin{vmatrix} 3 & 4 \\ -7 & 5 \end{vmatrix} = 3.5 - -7.4 = 15 + 28 = 43 \neq 0$$ A Solution exists and is unique!

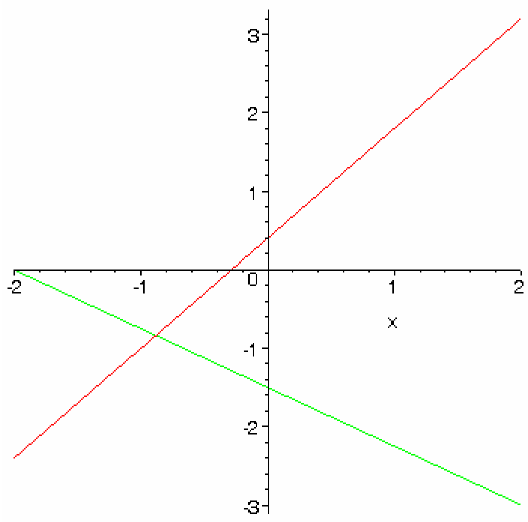
Now lets look at the two other determinants:

$$\Delta_x = \begin{vmatrix} -6 & 4 \\ 2 & 5 \end{vmatrix} = -30 - 8 = -38. \text{ and } \Delta_y = \begin{vmatrix} 3 & -6 \\ -7 & 2 \end{vmatrix} = 6 - 42 = -36$$

Solution: $x = \dfrac{-38}{43}$ and $y = \dfrac{-32}{43}$.

Let us take a look: the two equations can be written as lines in (x,y)

$$y = -\frac{3}{4}x - \frac{3}{2} \text{ and } y = \frac{7}{5}x + \frac{2}{5}$$



The lines meet where x and y are both negative and less than 1 in absolute value. x is closer to -1 than y.

Comparative Statics.

The main use of this is in economics is comparative statics.

Suppose we have two equations that determine an equilibrium with two endogenous variables and some exogenous variables

Example: Supply and demand.

$$x = D(p) + g$$
$$x = s(p)$$

Output is equal to demand and supply. What happens if g changes? Let us take the total differential of both

$$dx = D'dp + dg$$
$$dx = s'dp$$

Now, divide both equations by dg.

$$\frac{dx}{dg} = D'\frac{dp}{dg} + 1$$
$$\frac{dx}{dg} = s'\frac{dp}{dg}$$

Now, we can re-arrange this into a 2x2 system with the. Total derivatives $\frac{dx}{dg}, \frac{dp}{dg}$ as the variables to be solved for:

$$\frac{dx}{dg} - D'\frac{dp}{dg} = 1$$
$$\frac{dx}{dg} - s'\frac{dp}{dg} = 0$$

which can be written as $\begin{bmatrix} 1 & -D' \\ 1 & -s' \end{bmatrix}\begin{bmatrix} dx/dg \\ dp/dg \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

The Total Derivatives give us the effect on output and price of a change in the exogenous variable *g* (government demand?) given the supply and demand functions. Note that $D'<0$, $s'>0$.

Does a solution exist? $\Delta = -s' + D' < 0$, so that we know that there is a unique solution.

$$\Delta_x = -s' \Rightarrow \frac{dx}{dg} = \frac{\Delta_x}{\Delta} = \frac{-s'}{D' - s} > 0; \;\; \Delta_p = -1 \underset{4.12}{\Rightarrow} \frac{dp}{dg} = \frac{\Delta_p}{\Delta} = \frac{-1}{D' - s'} > 0$$

An increase in *g* boosts demand, so that both price and quantity increase. If *s*' is very big, then *dx/dg* almost 1 and *dp/dg* almost zero (horizontal supply curve): if D' almost 0 same (demand horizontal). Get zero Δ only if *s'=D'=0*. (both curves horizontal….).

Example: IS/LM.

$$Y = C(Y) + I(r) + g$$
$$M^s = L(r, Y)$$
$$1 > C' > 0 > I'; L_Y > 0 > L_r$$

Two equations, with two endogenous variables (*r, Y*) and two exogenous variables: M^s and g.

Total differential:

$$dY = C'dY + I'dr + dg$$
$$dM = L_Y dY + L_r dr$$

Now, to find out the effect of *dg* we set *dM=0* and divide through by *dg*.

$$\frac{dY}{dg}(1-C')-I'\frac{dr}{dg}=1$$

$$L_Y\frac{dY}{dg}+L_r\frac{dr}{dg}=0$$

If we re-arrange this, we get the familiar format:

$$\begin{bmatrix}1-C' & -I' \\ L_Y & L_r\end{bmatrix}\begin{bmatrix}dY/dg \\ dr/dg\end{bmatrix}=\begin{bmatrix}1 \\ 0\end{bmatrix}$$

This gives us the effect of an increase in government expenditure (shift in IS curve) on output and the interest rate.

Determinant $\Delta=(1-C')L_r+I'L_Y<0$. So, we can solve so long as the derivatives are not zero.

Cramer:

$\Delta_Y=L_r \Rightarrow \frac{dY}{dg}=\frac{L_r}{(1-C')L_r+I'L_Y}=\frac{(-)}{(-)}>0$ An increase in g raises output.

$$\Delta_r = L_Y \Rightarrow \frac{dr}{dg} = \frac{-L_Y}{(1-C')L_r + I'L_Y} = \frac{(-)}{(-)} > 0$$ It raises interest rate.

Can get the classical special cases by taking extreme limits. For example:

Liquidity Trap (horizontal LM curve): $L_r \to \infty \Rightarrow \frac{dy}{dg} = \frac{1}{1-C'}$ (no crowding out).

3x3 systems?

There is a useable formula for the determinant of a 3x3 Matrix. You can calculate the determinant by the following formula:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei + bfg + cdh - gec - hfa - idb$$

See the pattern? Then use cramer's rule.

Conclusion.

1. A non-zero determinant shows us a system of linear equations has a unique solution.

2. Cramer's rule enables us to solve a system of linear equations.

3. Comparative statics: enables us to evaluate the impact of a change in an exogenous variable on endogenous variables by linearizing the equilibrium equations.