INTRODUCTORY MATHEMATICS FOR ECONOMICS MSC'S.
LECTURE 5: DIFFERENCE EQUATIONS.

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SEPTEMEBER 2009.

Difference Equations.

Much of economics, particularly macroeconomics is *dynamic*. People save now to spend in the future: firms invest now to produce more in the future. Students learn now to earn more in the future ☺.

Difference equations treat time as discrete *t=1,2,3,……..T*.

First Order Difference Equation (FODE):

$$y_t = a + by_{t-1}$$

An Equilibrium is defined as a situation where *y* remains unchanged over time: $y_t = y_{t-1} = y^*$.

Hence: $y^* = a + by^* \Rightarrow y^* = \dfrac{a}{1-b}$

Hence, a unique equilibrium exists if and only if $b \neq 1$.

Questions: will we get to equilibrium if we start from some/all initial positions? If we get to equilibrium, will the path be monotonic? What happens if no equilibrium exists?

Solution of Difference Equations (linear FODE).

Suppose we start from an initial position y_0. Let us trace out what happens:

$$y_1 = a + by_0$$
$$y_2 = a + by_1 = a + b(a + by_0)$$
$$y_3 = a + by_2 = a + b(a + by_1) = a + ba + b^2(a + by_0) = a + ba + b^2 a + b^3 y_0$$
$$y_t = a + ba.....+ b^{t-1}a + b^t y_0$$

This is called the definite solution. Given an initial position, we have a clear sequence, a path of *y* traced out over time.

This simplifies.

If *b=1* (and hence no equilibrium exists), then the time path is

$$y_t = at + y_0$$

If $b \neq 1$? Well, let us first look at the case of a homogeneous FODE where *a=0*. In this case $y^* = 0$, and

$$y_t = b^t y_0$$

So, what happens as $t \to \infty$?

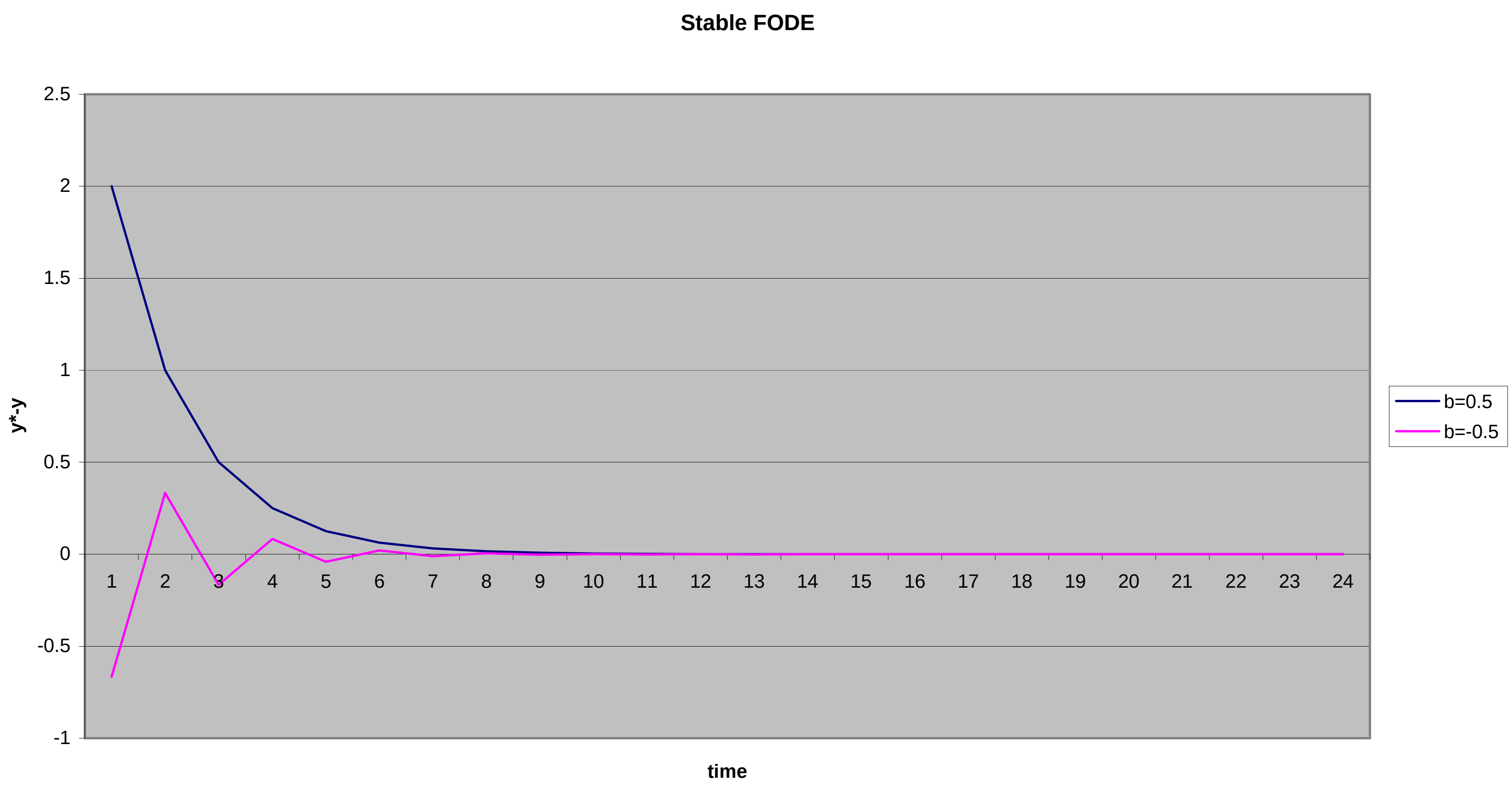
Case 1: If $0<b<1$ then $b^t \to 0$ monotonically.

Case 2: If $-1<b<0$ then $b^t \to 0$ but non-monotonically: it gets smaller in absolute value but alternates sign (negative numbers raised to an even number are positive, to an odd power are negative).

Case 3: $b=0$, y jumps to the equilibrium in period 1.

In all cases 1-3, we say that the difference equation is *globally stable* if the absolute value of b is less than 1, $-1<b<1$, or $|b|<1$. Wherever we start from, we end up at the steady-state equilibrium $y^*=0$.

Lets use an excel spreadsheet to take a look!



$$y_t = 2 + by_{t-1}$$

$$b = 0.5 \Rightarrow y^* = 4$$

$$b = -0.5 \Rightarrow y^* = 4/3$$

Case 4: $b>1$. In this case y goes to infinity (assuming initial y is positive).

Case 5: $b<-1$. In this case y goes to plus/minus infinity oscillating between – and +.

Now, it turns out that the stability of an FODE is only determined by the b constant. This is intuitive: the constant do not change, and so do not influence the dynamics.

General solution to the FODE $y_t = a + by_{t-1}$

If $b \neq 1$, then the general solution is

$$y_t = y^* + Ab^t$$

Where A is an arbitrary constant. This is determined by the initial condition (or in theory by a terminal condition – but in economics it is usually an *initial* condition!). To work this out we have at time 0

$$y_o = y^* + Ab^0 = y^* + A$$
$$\Rightarrow A = (y_0 - y^*).$$

Example: Cobweb model.

Supply in year t depends on price in year t-1:

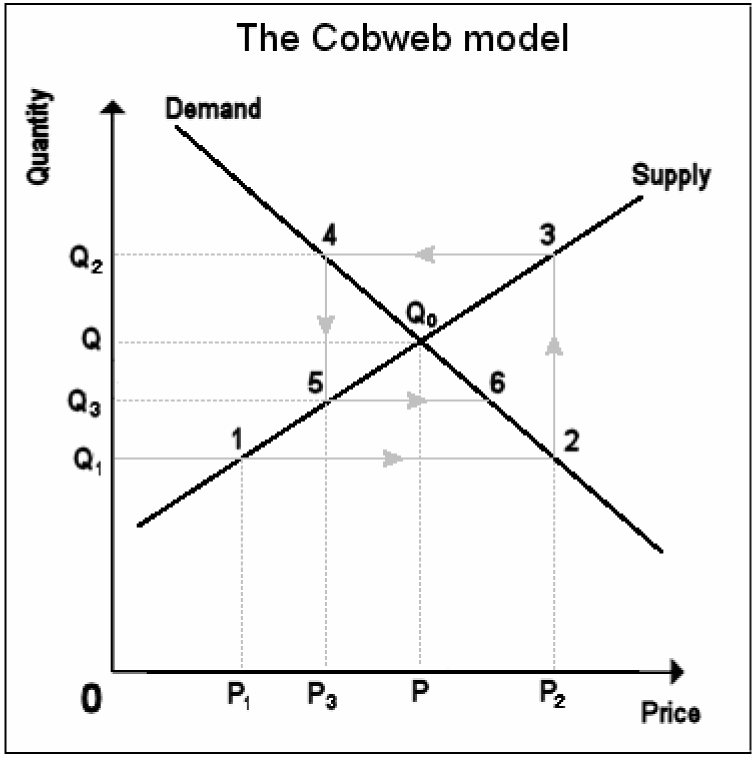
$$x_t = 1 - dP_t$$
$$x_t = sP_{t-1}$$

Supply equals demand: $$1 - dP_t = sP_{t-1} \Rightarrow P_t = -\frac{s}{d}P_{t-1} + \frac{1}{d}$$

Equilibrium: $P^* = \dfrac{1/d}{1+(s/d)} = \dfrac{1}{d+s}$

Is it stable? We have defined both *d,s* positive. Hence if $d<b$ (the demand curve is sloped less steeply than the supply curve) then the model is stable.

Since the coefficient is negative, the path of output will oscillate Around the equilibrium.



Note: Price on horizontal axis. At Q1, demand price is P2, so Q2 is s(P2), so demand price is P3…

Oscillates around equilibrium price and quantity P,Q.

This is stable, however no reason for the slope of demand to be more in absolute terms than slope of supply!

In that case you can get explosive oscillations.

Try it with a spreadsheet…..

The solution to the cobweb FODE is (assuming $-\frac{s}{d} \neq 1$)

$$P_t = (P_0 - P^*)b^t + P^*$$

$$P^* = \frac{1}{d+s}; \quad b = -\frac{s}{d}$$

Second Order Difference Equations.

These take the form

$$y_t + b_1 y_{t-1} + b_2 y_{t-2} = a$$

Now, the "long-run" equilibrium or particular solution is given by $y^* = y_t = y_{t-1} = y_{t-2}$. Two possibilities:

$$b_1 + b_2 \neq -1 \text{ and } b_1 + b_2 = -1$$

If $b_1 + b_2 \neq -1$, (the general case), then

$$y^* = \frac{a}{1 + b_1 + b_2}$$

If $b_1 + b_2 = -1$, (the special case) then we have two solutions (see Dowling). We will assume that the general case holds.

The solution is:

$$y_t = y^* + A_1 r_1^{\,t} + A_2 r_2^{\,t}$$

where

$$r_i = -\frac{b_1}{2} \pm \frac{\sqrt{b^2{}_1 - 4b_2}}{2}$$

If $b_1^2 > 4b_2$ Then there are two distinct real roots. If $b_1^2 = 4b_2$ then there is a unique root ($r_i = r = \frac{b_1}{2}$). If $b_1^2 < 4b_2$ there are not real roots, just imaginary or complex roots.

And A_i are arbitrary constants to be determined. Now, with a second order difference equation, you need 2 initial conditions: (y_0, y_1), and these will determine the two arbitrary constants.

Note: if we compare the solutions to a first order difference equation: when r_i, the two solutions become identical (check: if $b_2 = 0$,

$$r_i = -\frac{b_1}{2} \pm \frac{\sqrt{b^2{}_1}}{2} = -b_1, 0.$$

Note: the negative sign on the *b* is there because of the way we wrote the equations.

- Stability is dertermined by the values of the roots[1] r_i. Let assume that there are two distinct real roots.

[1] The roots of the equation are called *eigenvalues*: in general, if there is an *n*th order difference equation, the solution will be represented by a linear function of the *n* roots or eigenvalues.

$$y_t = y^* + A_1 r_1^{\,t} + A_2 r_2^{\,t}$$

- If $-1 < r_i < 1$ (both roots less than 1 in absolute size) we have global convergence: wherever you start from, you get to the equilibrium.

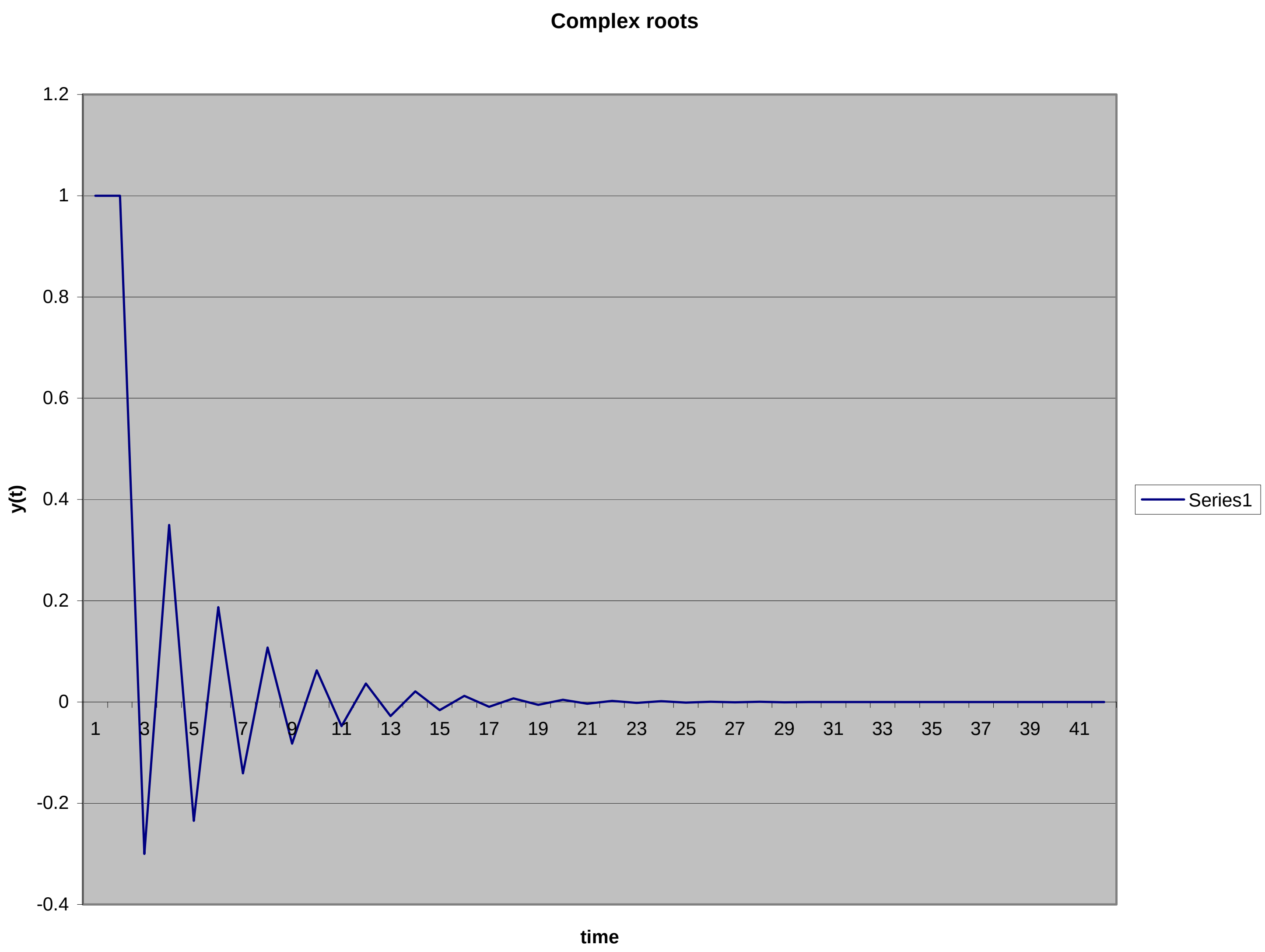
- If $|r_1| < 1, |r_2| > 1$ (one stable, one unstable) you have *saddle-point stability*. For initial conditions where the unstable root has a zero weight ($A_2 = 0$), you proceed to equilibrium. Otherwise you explode.

In economics, we often use saddle-path solutions. That is because there is a unique path to equilibrium, which is traced out by the negative root.

What if the roots are not real (imaginary): when $b_1^{\,2} < 4b_2$. I will not go into the details here: however, basically you get cycles in this case. These can be explosive or convergent. It is easiest to see an example.

$$y_t - 0.5y_{t-1} + 0.2y_{t-2} = 0$$
$$y^* = 0; y_0 = y_1 = 1.$$

Using excel spreadsheet. In a column (row). Type in the initial values 1,1 in the first to rows (let us say y_o=C1=1, y_1 =C2=1). Then type in "=-0.5*C2+0.2*C1: then make chart….

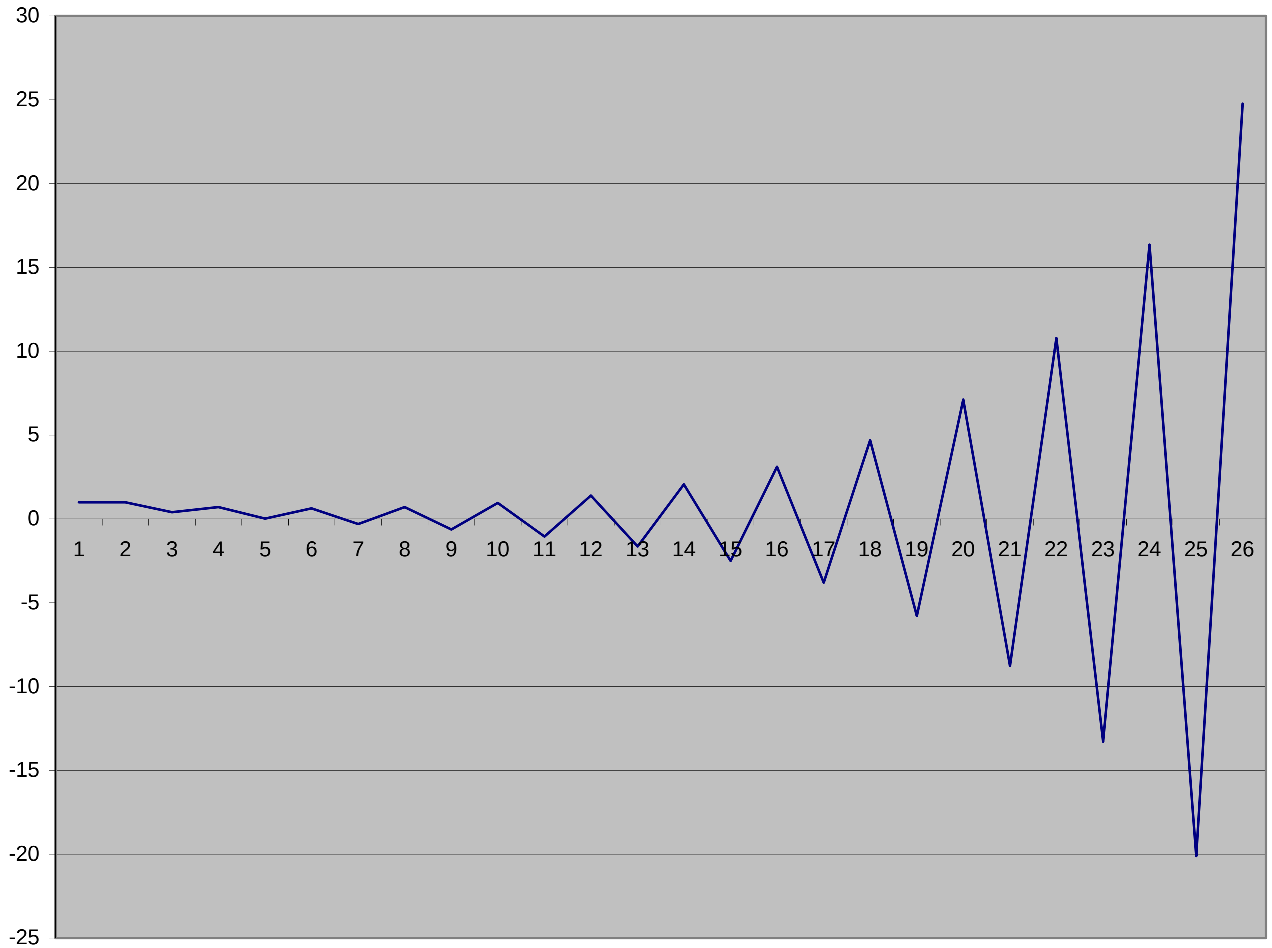

Complex roots
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
y(t)
1 3 5 7 9 11 13 15 17 19 21 23 25 27 29 31 33 35 37 39 41
time
Series1

Try:
$b_1 = -0.1; b_2 = 0.9:$

Here we have explosive oscillations!

You can use a spreadsheet to see what happens to a second order difference equation.

With complex roots, you get oscillations, which can either tend to the equilibrium, explode, or just cycle and never converge.



Conclusion.

1. Linear difference equations: express the current value as a linear function of past (lagged) values.

2. The equilibrium solution is derived by setting the current and lagged values equal.

3. The stability of the difference equation is determined by the coefficients on current and lagged values (*not* the constant). These determine the *roots* or *eigenvalues* of the difference equation.

4. A root is stable if it is less than one in absolute value. If all roots are stable, then the difference equation is globally stable: wherever you start from you will converge to the equilibrium. If the root is positive, you will converge monotonically, if negative it will oscillate.

5. With a second (or higher) order difference equation, you can get complex roots which can give cycles.