Economics Letters 16 (1984) 205–212
North-Holland

THE EXISTENCE OF MIXED-STRATEGY EQUILIBRIA IN A PRICE-SETTING OLIGOPOLY WITH CONVEX COSTS

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Received 15 March 1984

This paper shows that a mixed-strategy equilibrium will exist in an industry producing a homogeneous product with perfectly informed consumers if firms are identical, and have weakly-convex costs. This generalises an earlier result by Dasgupta and Maskin.

Dasgupta and Maskin (1981,1982) have provided a very general framework for demonstrating the existence of mixed-strategy equilibria in games with discontinuous payoffs. They have applied this to several well-known economic models, including the Bertrand–Edgeworth price-setting duopoly (where there are perfectly informed consumers, and firms have constant – or zero – average costs of production up to capacity). Perhaps as a consequence, there has been a considerable revival of interest in mixed-strategy solutions to Bertrand–Edgeworth models [see Allen and Hellwig (1983), Brock and Scheinkman (1983), Kreps and Scheinkman (1983), inter alia]. These models have employed the simplest case of constant average costs up to capacity, which is a very restrictive assumption indeed. Can we generalize this assumption to the more plausible case of weakly-convex cost functions? The answer given in this paper is that the D–M theorem can be applied in the case of convex cost functions, but only in the case where firms are identical, and when firms setting the same price have the same demand. This result is important not only because of the greater attractiveness of convex cost functions, but

* I am grateful to Jim Mirrlees and Ben Lockwood for their comments.

also because it brings out the limits of the D–M theorem more clearly than the simpler Edgeworthian case.

We shall first set up the framework for a price-setting duopoly where firms have identical convex costs [this framework originated in Shubik (1959, ch. 5) and Shapley (1957)]. We only consider the case of duopoly, but the results obviously generalize.

A.1. Firms $i = 1, 2$ have the same total cost function $c(x_i)$ which is continuous, strictly increasing, and (weakly) convex in output x_i.

A.2. Industry demand $F: R_+ \to R_+$ is a continuous function, which is bounded from above by some $K > 0$. There is a choke-off price $\bar{p}$ such that $F(p) = 0$ for $p \geqslant \bar{p}$.

Given A.1–A.2, we can define the firms' supply function,

$$s(p) = \underset{x_i \in [0, K]}{\arg\max}\, p \cdot x_i - c(x_i).$$

We impose $x_i \leqslant K$ since this makes $s(p)$ well defined even in the case of constant returns, and the restriction is valid because we know that neither firm can sell more than K. Firms set prices p_i, $i = 1, 2$, and then produce to order. There is then voluntary trading, so that the quantity sold by firm i is the minimum of its supply $s(p_i)$, and the demand for its output.

Demand for each firm is a function of the prices set, $d_i(P)$, where $P = (p_1, p_2)$. Assuming that customers are perfectly informed, and adopting the convention of 'equal shares' if $p_1 = p_2$, we have

$$d_i(P) = F(p_i)/2, \qquad p_i = p_j,$$
$$d_i(P) = F(p_i), \qquad p_i < p_j.$$

If $p_i > p_j$, we can specify contingent demand as either

$$d_i(P) = \max\left[0, F(p_i) - s(p_j)\right], \quad \text{or} \tag{S}$$

$$d_i(P) = \max\left[0, 1 - \left(F(p_j)/s(p_j)\right)\right]. \tag{E}$$

Specification S originates in Levitan and Shubik (1972), E in Edgeworth

(1925). The results in this paper hold for both specifications.

The firms' payoffs are given by: $\Pi_i: A \to R_+$ ($A = [0, \bar{p}]^2$),

$$\Pi_i(P) = p_i \cdot x_i - c(x_i), \quad \text{where} \tag{1}$$

$$x_i = \min[s(p_i), d_i(p_i)].$$

We can truncate each player's strategy set A_i to $[0, \bar{p}]$ by A.2. Note that the payoff functions $\Pi_i: A \to R_+$ are continuous in $P \in A$, except on a subset $\mathring{A}$ of A, where

$$\mathring{A} = \{P \in A \mid p_i = p_j\}.$$

Dasgupta and Maskin's (1981, p. 36; 1982, p. 5) theorem gives sufficient conditions for the existence of a mixed-strategy equilibrium in the game $[[0, \bar{p}], \Pi_i: i = 1, 2]$. See the appendix for a statement of the theorem. The only conditions which do not obviously hold are:

(A) $\Sigma \Pi_i$ is upper-semicontinuous (u.s.c.) in P. That is, if the sequence $\{P_n\}$ tends to P, then

$$\limsup_{P_n \to P} \sum \Pi_i(P_n) \leqslant \sum \Pi_i(P).$$

(B) Π_i is left-lower semicontinuous in p_i. That is, if $\{p_{in}\}$ tends to p_i from below, then

$$\liminf_{p_{in} \to p_i^-} \Pi_i(p_{in}, p_j) \geqslant \Pi_i(p_i, p_j).$$

D–M (1982, pp. 8–13) show that condition (B) presents no problem in this model. As D–M argue, since customers are perfectly informed, the only discontinuities in the firm's payoff function will occur when $p_i = p_j$, and as we consider the firm raising its price, it will face discontinuous falls in demand and profits. In our theorem, we show that, under A.1–A.2, condition (A) is also satisfied, so that we know that a mixed-strategy equilibrium exists.

Theorem. A mixed-strategy equilibrium exists in $[A_i, \Pi_i; i = 1, 2]$ *under both the E and the S specifications of contingent demand.*

Proof. From (1),

$$\sum_{i=1,2} \Pi_i(P) = \sum p_i \cdot x_i - \sum c(x_i), \quad \text{where}$$

$$x_i = \min[s(p_i), d_i(P)].$$

Any violation of upper-semicontinuity in total profits can only occur due to discontinuities in total costs $\Sigma c(x_i)$ as P varies. Total revenue $\Sigma p_i \cdot x_i$ is continuous in P, since [as D–M (1982, p. 13) argue] the discontinuous shifts in clientele from one firm to the other occur when both firms set the same price, and hence derive the same profit per customer.

We now show that $-\Sigma c_i(x_i)$, and hence total profits, are u.s.c. in P.

Consider any sequence $\{P_n\}$ such that $P_n \to P$. The sequence $\{P_n\}$ will generate corresponding sequences for the outputs of firms 1 and 2, denoted $\{x_{in}\}$, and for the *total* output of the industry, denoted $\{x_n\}$. We now show that, under either the E or the S specification of contingent demand, total output x is continuous in P, even though the individual outputs are not. Under the S specification this is obvious. If we define p_m as the higher of the two prices, $p_m = \max_{i=1,2}[p_i]$, then under S,

$$x = x_1 + x_2 = \min[F(p_m), s(p_1) + s(p_2)]. \tag{2}$$

Given that under S total output x is continuous in P, it is simple to show that condition (A) holds, that total profits are u.s.c. in P. Since x is continuous in P, if $P_n \to P$, and $x_n \to x$, then *at* P the actual output x_0 is equal to the limiting output x. If $P_n \to P$ where $P \in A - \mathring{A}$, then by the continuity of c,

$$\lim_{P_n \to P} \sum_i c(x_{in}) = \sum c(x_i), \tag{3}$$

where x_i is the limit of $\{x_{in}\}$. If, however, $P_n \to P$, where $P \in \mathring{A}$, then by the convexity of costs

$$\lim_{P_n \to P} \sum_i c(x_{in}) \geqslant 2 \cdot c(x/2), \tag{4}$$

where x is the limit of $\{x_n\}$. Hence, combining (2) and (3),

$$\lim_{P_n \to P} \sum_i \Pi_i(P_n) \leqslant \sum \Pi_i(P).$$

The essence of the proof is that at the point of discontinuity P where $p_i = p_j$, under the 'equal shares' assumption the firms switch from producing unequal outputs to producing equal outputs. Since costs are identical and convex, costs are minimized in the neighbourhood of P. In the case of strictly-convex costs the inequalities in the proof will be strict, resulting in the relationship between p_i and total profits as depicted in fig. 1.

Under E, the argument is a little less straightforward. Since firms are identical, it is more convenient to consider the sequences of outputs and prices where the indexes k and m refer to the firm setting the lowest price

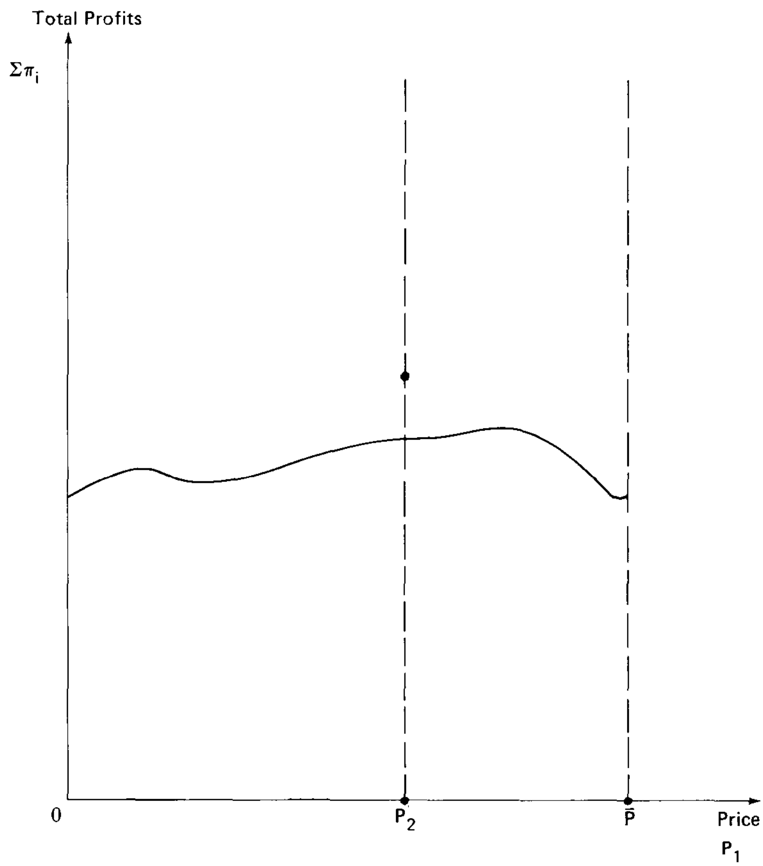

Fig. 1. The upper-semicontinuity of total profits in p_1.

and the firm setting the highest price, respectively. Thus we define the sequence $\{p_{kn}\}$ with $p_{kn} = \min[p_{2n}, p_{1n}]$, and $p_{mn} = \max[p_{1n}, p_{2n}]$, and the corresponding sequences of the firms' outputs $\{x_{kn}\}$ and $\{x_{mn}\}$. Note that the total output sequence $\{x_n\}$ is unaffected by this re-indexing. We have to show that x is continuous in P. The difficulty comes because it is not immediately clear that if $P_n \to P$, where $P \in \mathring{A}$, the total output might not 'jump'. To demonstrate that x is continuous in P, we need to consider two cases of $\{P_n\}$.

Case 1. There exists n_0 such that for $n > n_0$, $p_{kn} < p_{mn}$. That is, for some subsequence of $\{P_n\}$, the two firms set different prices, $P_n \in A - \mathring{A}$. In this case, firms' outputs are given by

$$x_{kn} = \min[s(p_{kn}), F(p_{kn})], \tag{5}$$

$$x_{mn} = \min[(1 - x_{kn}/F(p_{kn})) \cdot F(p_{mn}), s(p_{mn})], \tag{6}$$

$$x_n = x_{kn} + x_{mn}.$$

Since p_{kn} and p_{mn} vary continuously with P_n, from (5) and (6) we can see that x_n is continuous in P for $P \in A - \mathring{A}$. However, if $P_n \to P$, where $P \in \mathring{A}$, we have a 'regime-switch', where for $n > n_0$ x_n is given by (5) and (6), but at the limit outputs are equal. We denote the output at the limit $x_0 = F(p)$. We need to show that as $P_n \to P$, where $p_1 = p_2 = p$, $x_n \to x_0$.

Evaluating the limits of x_{kn} and x_{mn},

$$x_{kn} \to \min[s(p), F(p)] =_{\text{def}} x_k,$$

$$x_{mn} \to \min[(1 - x_k/F(p)) \cdot F(p), s(p)] =_{\text{def}} x_m.$$

If $x_k = F(p)$, then $x_m = 0$, so that $x_k + x_m = F(p)$, which is equal to total output at P [since $x_k = F(p)$ implies $s(p) \geqslant F(p)$]. If

$$x_k = s(p) \leqslant F(p) \quad \text{so that} \quad x_m \geqslant 0, \quad \text{then}$$

$$x_m = \min[F(p) - s(p), s(p)]. \quad \text{Hence}$$

$$x_k + x_m = \min[F(P), 2 \cdot s(P)] = x_0.$$

Thus the limit of total output as $P_n \to P$ is equal to the output at the limit.

Case 2. There exists n_0 such that for $n > n_0$, $p_{1n} = p_{mn} = p_n$. That is, $P \in \mathring{A}$. Here we can immediately see that the value of x at P will equal the limit of $\{x_n\}$ since if both firms set the same price

$$x_n = \min[F(p_n), 2 \cdot s(p_n)],$$

$$x_k = s(P),$$

$$x_m = \min[F(P) - s(P), s(P)].$$

Thus the limit of total output as $P_n \to P$ is equal to the output at the limit.

Thus in both cases, the industry output x is continuous in prices P for the E specification. Given the 'equal shares' assumption, eqs. (3) and (4) will go through as before, thus establishing the theorem. Q.E.D.

This proof brings out the crucial role played by the assumption that the firms are *identical*, coupled with the assumption that firms face the same demand if they set the same price. In the Bertrand and Edgeworth games studied by D–M, neither of these considerations is crucial, since firms have constant average costs, so that profit per customer is the same for both firms when $p_i = p_j$. Thus firms need not be identical in the sense of having the same capacity, and demand need not be shared equally when $p_i = p_j$. D–M's proof would still go through with the general condition that when $p_i = p_j$,

$$d_i(P) + d_j(P) = F(P).$$

Under the more general assumption of convex costs, both assumptions are crucial, since without them we cannot ensure that total industry costs suddenly fall (and hence total profits rise) when prices are equal.

Appendix

Theorem [*Dasgupta and Maskin (1981)*]. *For all agents i, let the strategy space $A_i \in R^m$ ($m \geqslant 1$) be non-empty and compact, and let Π_i be continuous except on a subset $\mathring{A}$ of A, where*

$$\mathring{A} = \{P \in A \mid p_i = p_j, i \neq j\}.$$

If (A) $\Sigma \Pi_i$ *is u.s.c. in P for* $P \in A$, *and* (B) Π_i *is bounded and left-l.s.c in* p_i, *then* $[(A_i, U_i);\ i = 1 \ldots n]$ *possesses a mixed-strategy equilibrium.*

[D–M (1982, p. 5) require an even weaker continuity condition to (B), and allow for a more general class of sets than $\mathring{A}$.]

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