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BERTRAND–EDGEWORTH EQUILIBRIA WHEN FIRMS AVOID TURNING CUSTOMERS AWAY*

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This paper provides a simple solution to the problem of non-existence of pure-strategy equilibria in Bertrand–Edgeworth models with strictly convex costs. The voluntary-trading constraint in standard Bertrand–Edgeworth models is generalized to allow for there being costs incurred when customers are turned away. So long as the industry is sufficiently large, the presence of such costs ensures that the competitive price will be an equilibrium. There will be other single price equilibria, but if the offence costs are small, all equilibria will be close to the competitive outcome.

I. INTRODUCTION

IN MODELS of price-setting oligopoly, it is usually assumed that firms set prices, and then given these prices trade occurs. There have traditionally been two approaches to the trading process: the first, originating in Chamberlin [1933], assumes that the output of firms is equal to consumer demand at that price; the second, originating in Edgeworth [1897, 1922], assumes that output is determined by a voluntary trading constraint, namely that output is the smaller of demand and the firm's profit maximizing output at the price set. In Bertrand–Edgeworth models with a homogeneous product, perfectly informed consumers and strictly convex costs, there is a general problem of non-existence of a pure-strategy equilibrium, resulting from the discontinuities in demand and the voluntary-trading constraint, (see, for example, Shubik [1959], Dixon [1987a, Theorem 1]).[1] This paper suggests a generalization of the voluntary trading constraint in standard Bertrand–Edgeworth models which with many firms will ensure (a) that the competitive price is an equilibrium (Theorem 1), and (b) that any equilibria that exist are close to the competitive equilibrium (Theorem 2).

The idea is very simple, and rests on firms facing a cost to turning customers away. In the Chamberlinian framework, firms never turn customers away, which implicitly assumes that it is very costly to do so; in Bertrand–Edgeworth models, it is assumed that it is *costless* to turn customers away. In this paper we assume that there may be some cost to turning customers away. Initially we assume there to be a lump sum cost of

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[1] For a comparison of these approaches with differentiated products, see Benassy [1989].

$\Upsilon \geqslant 0$ to turning customers away: the standard Bertrand–Edgeworth model is a special case where $\Upsilon = 0$. It is very plausible to assume that there is some cost to failing to meet demand, in terms of loss of goodwill, reputation, or offence caused. The presence of such costs is absolutely standard in Operations Research and inventory models (see, for example, Taha [1982, ch. 13] for textbook treatment and references). This means that firms are willing to meet demand in excess of their profit-maximizing supply in order to avoid incurring customer offence costs Υ. It is crucial that even small offence costs $\Upsilon > 0$ can lead to a large increase in supply: this is because at the profit-maximizing output the firms objective function is flat, so that an increase in output has no first-order effect. The generalization of the Bertrand–Edgeworth voluntary trading constraint to allow for costs to be incurred when customers are turned away is enough to ensure that the competitive price is an equilibrium when the industry is sufficiently large.

The intuition behind the existence result is very simple. In the absence of there being any costs to turning customers away, the competitive price is not an equilibrium: if any firm raises its price, then there will be an excess demand for the lower priced firms, who will not extend their output beyond their supply function. Hence the firm raising its price will face some residual demand, and will wish to raise its price above marginal cost (except in special cases where industry demand is horizontal at the competitive price, or there is some discontinuity in marginal cost). The presence of customer offence costs Υ will encourage firms to raise their output beyond their usual supply function, hence reducing and possibly eliminating the residual demand (and hence profits) of any firm raising its price.

The assumption of lump sum costs is merely a convenient expositional simplification, and is in no way crucial to the results. In Theorem 3, we allow for the loss of goodwill to vary with the level of unsatisfied demand, e. So long as the marginal cost of unsatisfied demand is positive at 0, (i.e. $\Upsilon = \Upsilon(e)$, $\Upsilon'(0) > 0$), the competitive price is an equilibrium if the industry is large enough. This is a very weak condition, and reflects the fact that when firms are on their supply functions, there is no first order loss of profits in expanding output: therefore output may expand beyond normal supply to avoid a first-order cost, which is enough to establish the result.

The framework and results of this paper are closest to Dixon [1987a]. That paper considered the existence of epsilon-equilibria in price-setting with the standard Bertrand–Edgeworth voluntary trade constraint: this paper considers the case of a strict Nash equilibrium in prices (epsilon $= 0$), with the generalized voluntary trading constraint to take account of customer offence costs. The solution to non-existence put forward in this paper is in many ways more attractive than that in Dixon [1987a]. First, the result that even small menu costs can lead to the competitive price being an equilibrium only applied to a particular specification of contingent demand [1987a, Theorem 2 and Proposition 1]. This paper makes very general assumptions about contingent demand that apply to all of the standard specifications. As such, this paper is the only general result establishing the competitive price as an equilibrium with contingent demand generated by First-come-first-served rationing. Second, although the argument here relies on replication, the fact that even small customer offence costs can lead to large increases in supply means that the industry does not need to have many firms to be "large". Thirdly, we can characterize the complete set of equilibria in this paper (see Theorem 2), since all equilibria involve firms setting the same price, whereas in Dixon [1987a, Theorem 3] we were merely able to characterize the closeness of episilon-equilibria to the competitive price. For a summary of other approximation results see Dixon [1987a, p. 49]. In section II of the paper we outline the basic framework and assumptions: in section III we state and prove the results.

II. THE BERTRAND–EDGEWORTH FRAMEWORK WITH CUSTOMER OFFENCE

This paper adopts similar assumptions to Dixon [1987a]. However, for ease of exposition we have restricted the generality of the assumptions to the most interesting cases. Without replication, there is a set of n identical firms producing a homogeneous product. If the industry is replicated r times there are rn identical firms. Each firm $i = 1 \dots rn$ can set its own price $P_i \varepsilon [0, \infty)$, the rn vector of prices being $\underline{P}$.

Assumption 1 (A1). Costs. Each firm has a total cost function $C: R_+ \to R_+$, which is strictly increasing, continuously differentiable, and strictly convex in output x_i.

This enables us to define the supply function $S: R_+ \to R_+$ where:

$$S(p) \equiv \underset{x_i}{\operatorname{argmax}}\, p \cdot x_i - c(x_i)$$

Note that under (A1), S is a continuous strictly monotonic function. The corresponding profit function is $\xi(p)$:

$$\xi(p) \equiv p \cdot S(p) - c(S(p))$$

The supply function gives the profit maximizing output at any given price. We now assume that there is a fixed lump sum cost Υ to turning customers away. This can be seen as arising because of the offence or frustration caused to customers when they are unable to buy all they would like from a particular firm at a posted price ('loss of goodwill'). In the Chamberlinian framework, where customers are never turned away, Υ is seen as being very large. In the standard Bertrand–Edgeworth framework, $\Upsilon = 0$, and firms will turn customers away if demand exceeds supply as defined by $S(p)$. However, the Bertrand–Edgeworth assumption is in some way as extreme as the Chamberlinian: although we do not model it formally here, it seems reasonable that there will be some loss of reputation caused by failure to meet demand. Shopkeepers and firms do not like to turn away customers empty-handed. The crucial point is that even if the costs of offence are small, they may still have a significant effect on the willingness of the firm to supply more output than $S(p)$. This is because at $S(p)$ there is no first order effect on profits of an increase in output: depending on the convexity of the cost function, even small offence costs can lead to a large increase in supply forthcoming from a firm.

In order to analyze the effect of offence cost we will define the largest output at each price that the firm is willing to supply to avoid incurring Υ. This is given by the augmented supply function σ:

$$\sigma(P) \equiv \max\{x: p\cdot x - c(x) \geqslant \xi(p) - \Upsilon\}$$

The extra output that can be elicited by the avoidance of customer offence is $\sigma(p) - S(p)$. For $\Upsilon > 0$, this is strictly positive. To illustrate the notion that even small offence costs can have an important effect, consider the cost function $c(x) = cx^2/2$. In this case, $\sigma(p) - s(p) = +\sqrt{(2\cdot\Upsilon/c)}$. For any given $\Upsilon > 0$, the extra output can be large if c is small (the 'less convex' is $c(x)$). If demand exceeds $\sigma(p)$, then it is not worth the firm meeting demand, and it will prefer to incur offence cost Υ and produce at $S(p)$, earning profit $\xi(p) - \Upsilon$.

Turning to the demand side, the industry demand function gives the demand when all firms set the same price:

Assumption 2 (A2). Industry Demand. With r replications, industry demand is given by $rF(p)$ where:

(a) $F(0) = K$ where K is strictly positive and bounded
(b) There exists $\bar{p}$ such that for all $p \geqslant \bar{p}$, $F(p) = 0$.
(c) F is continuous, strictly decreasing and differentiable on $[0, \bar{p})$.

Assumptions (A1) and (A2) are not as general as they could be. In particular, it would be simple to generalize (A1) to weak convexity so that it encompasses the Bertrand model. However, if c has affine segments, then s is a correspondence, the firm being indifferent between outputs. The presence of offence costs $\Upsilon > 0$ is irrelevant in this case, since there is no cost to meeting additional demand. The assumption of strictly convex costs focuses on the important cases. We are now in position to define the competitive price θ:

$$(1) \qquad F(\theta) = nS(\theta)$$

To ensure that θ is uniquely defined, and to avoid trivialities, we assume $S(\bar{p}) > 0$, so that $\bar{p} > \theta > 0$. Note that replication does not alter θ (replication multiples both sides of (1) by r).

We now come to the trading process and the specification of the contingent demand function. The contingent demand curve for firm i gives the demand for its output as a function of the rn prices set:

Definition: Contingent Demand d_{ri}: $R^{rn} \to R_+$, $\quad d_{ri} = d_{ri}(p)$

We have discussed the specification of contingent demand in Bertrand–Edgeworth models in Dixon [1987a], and more generally in Dixon [1987b]. Let us first define the output x_i produced by firm i setting price p_i *given* the level of demand d_{ri}:

$$(2) \qquad x_i = \begin{cases} d_{ri} & d_{ri} \leqslant \sigma(p_i) \\ S(p_i) & d_{ri} > \sigma(p_i) \end{cases}$$

Equation (2) generalizes the voluntary trading constraint, and says that the firm will meet demand up to the augmented supply curve σ. If demand

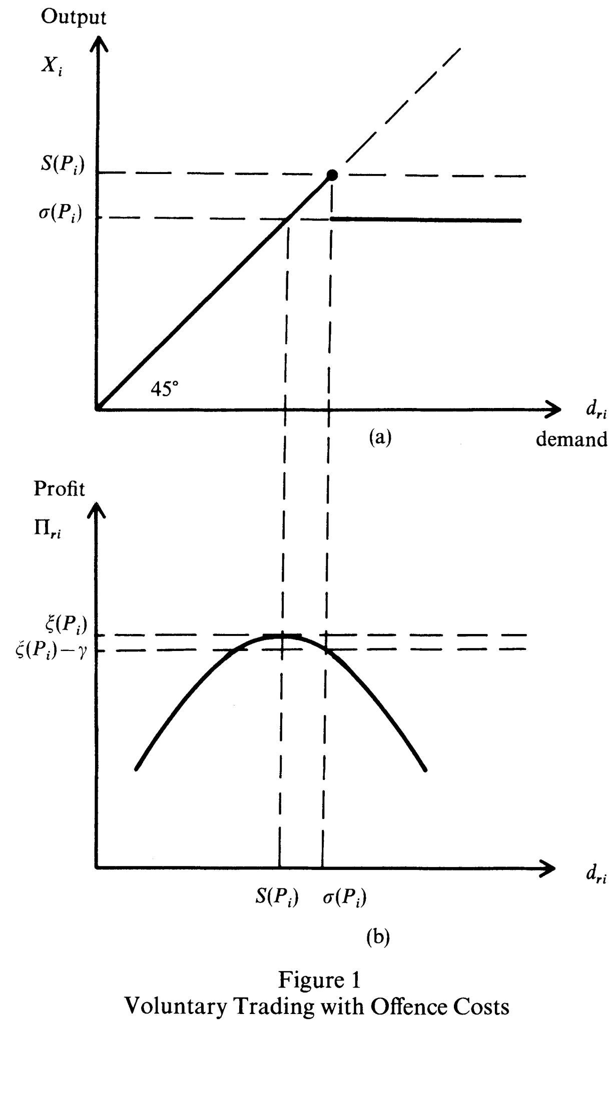

Figure 1
Voluntary Trading with Offence Costs

exceeds $\sigma(p_i)$, it will pay the firm to turn away customers and reduce output to $S(p_i)$. This is depicted in Figure 1a, with demand on the horizontal axis, and output on the vertical: in Figure 1b, the corresponding profits are given. With $\Upsilon = 0$, (2) reverts to the standard "min" condition for voluntary trading in Bertrand–Edgeworth models ($x_i = \min[S(p_i), d_{ri}]$). Here, trade can occur at outputs greater than $S(P_i)$, because the firm is willing to expand production over the profit maximizing supply to avoid offending customers. However, note that the underlying trading process is exactly the same as in the Bertrand–Edgeworth model: the firms set prices $\underline{p}$, and then voluntary trading occurs given those prices.

Given the trading process, there are different ways of specifying contingent demand (see Dixon [1987a], [1987b]). For the results of this paper, we only need to specify very general properties of the contingent demand functions d_{ri}, which are satisfied by the two most common specifications (CCD and FCFS in Dixon [1987a, p. 53]. As we show below, all equilibria are single-price equilibria, so we will be interested in contingent demand of firm i when all other firms set the same price, which we will write as $d_{ri}(p_i, p)$ for shorthand.

Assumption 3 (A3). Contingent Demand

P1: For all $p_i \geqslant 0, \quad d_{ri}(p) \leqslant rF(p_i)$.
P2: If p_i is the lowest price ($p_i < p_j$ for all $j \neq i$): $d_{ri}(p_i, p_{-i}) = rF(p_i)$
P3: If $p_i = p_j, \quad d_{ri} = d_{rj}$.
P4: $d_{ri}(p_i, p_{-i}) = 0$ if for some j $p_j < p_i$ and $d_{rj} = x_j$.
P5: d_{ri} is continuous except when there exists j such that $p_j = p_i$.
P6: d_{ri} is strictly decreasing when positive and continuous.

$$\text{P7: } \lim_{\varepsilon \to 0} d_{ri}(p+\varepsilon, p) = \begin{cases} 0 & (rn-1)\sigma(p) \geqslant F(p) \\ rF(p) - (rn-1)S(p) & \text{otherwise} \end{cases}$$

Properties P1–P6 are completely standard: P3 is an equal shares assumption; P4 says that a lower priced firm j meets all demand, then higher priced firms will have no demand. Note that unlike Bertrand–Edgeworth models, we can not say that d_{ri} is everywhere decreasing: this is because at points of discontinuity, a small rise in price p_j may trigger firms still at the initial price back to their supply $S(p)$ if demand for their output is increased above $\sigma(p)$. Property 7 merely states that if lower priced firms meet all demand (i.e. $(rn-1)\sigma(p) \geqslant F(p)$), then $d_{ri} = 0$ for prices $p_i > p$. If lower priced firms do not meet demand ($(rn-1)\sigma(p) < F(p)$), then the limit of d_{ri} as p_i tends to p from above is industry demand less the supplies of lower priced firms (i.e. $F(p) - (rn-1)S(p)$). Again, this property is satisfied by all standard specifications of contingent demand. The most common specification of contingent demand is variously called parallel/efficient rationing or Compensated contingent demand (CCD). In the presence of excess demand, one can imagine identical consumers being rationed on an 'equal shares' basis; in the absence of income effects (or with compensation), the residual demand for higher priced firms is then industry demand less the output of lower priced firms. In the case of duopoly, we would have:

$$d_1(p_1, p_1) = \begin{cases} F(p_1) - S(p_2) & p_1 > p_2 \\ F(p_1)/2 & p_1 = p_2 \\ F(p_1) & p_1 < p_2 \end{cases}$$

It can be verified that Properties P1–P7 of Assumption 3 are satisfied.

We are now in a position to define the Payoff function for each firm, giving profits as a function of prices set $\underline{p}$ given contingent demand function $d_{rj}(\underline{p})$:

Definition: Payoff function $\Pi_{rj}\colon R^{rn} \to R$

$$\Pi_{rj}(\underline{p}) = \begin{cases} \xi(p_j) - \Upsilon & d_{rj}(\underline{p}) > \sigma(p_j) \\ p_j \cdot d_{rj}(\underline{p}) - c(d_{rj}(\underline{p})) & \text{otherwise} \end{cases}$$

We will be considering the existence of pure-strategy Nash-equilibria in the game $[[0, \bar{p}], \Pi_{rj}\colon j = 1 \ldots rn]$. First, however, it is useful to recall existing results in this framework when $\Upsilon = 0$: (a) Under A1–A3, no equilibrium in pure strategies exists (Dixon, [1987a, p. 54]); (b) Under A1–A3, an equilibrium in mixed-strategies exists (Dixon [1984, p. 207]), and Maskin [1986], based on Dasgupta and Maskin [1987a, b]).

That no equilibrium in pure strategies exists is not a recent result: it dates back to Edgeworth's studies of his cycles [1897] and [1922]. The reason behind the non-existence can easily be illustrated when we note from the definition of θ and A3–P7 that d_{ri} are right-continuous around $\underline{\theta}$. At the competitive equilibrium, each firm produces at price equal to marginal cost, $x_i = S(\theta)$, and $\Pi_{ri} = \xi(\theta)$. If a Nash-deviant raises its price from $\underline{\theta}$, its change in profits is:

$$\partial \Pi_{ri}/\partial p_i = S(\theta) + (\theta - c'(S(\theta)) \cdot \partial d_{ri}/\partial p_i$$

Since price equals marginal cost at the competitive outcome, $\theta = c'$; so long as $\partial d_{ri}/\partial p_i$ is bounded (i.e. demand is not horizontal), the second RHS term is zero so that $\partial \Pi_{ri}/\partial p_i = S(\theta) > 0$. Since there is no first order loss to a reduction in output, it always pays the Nash-deviant to raise its price above θ, so that the competitive outcome is not an equilibrium (see Dixon 1987a, Theorem 1). As is well known, the only possible pure-strategy equilibrium in Bertrand–Edgeworth models is the competitive outcome (Shubik [1959, p. 100, Theorem 2]).

This non-existence result does pose a genuine problem. One response is to employ different assumptions about cost to A1 to guarantee the existence of a pure-strategy equilibrium: usually by assuming constant average/marginal cost (as in the Bertrand case). The alternative is to use mixed-strategy equilibria. These are open to questions of plausibility. However, the most serious problem with mixed-strategy equilibria in Bertrand–Edgeworth models is their precise characterization. Despite much recent work on this (Allen and Hellwig [1986], Kreps and Scheinkman [1983] *inter alia*), at present we do not know what a mixed-strategy equilibrium looks like outside of the Edgeworthian world of constant unit cost up to capacity. Certainly, no one has managed to say much about the class of mixed-strategy equilibria corresponding to the strictly convex and continuously differentiable cost functions of A1. In this paper, in contrast, we can establish not only the existence of equilibria, but also characterize the full set of equilibria (at least in a large industry).

III. THE EXISTENCE AND CHARACTERIZATION OF EQUILIBRIA

Our first result concerns the competitive price θ: for given $\Upsilon > 0$ if the economy is large enough, then $\underline{\theta}$ is an equilibrium in $[[0, \bar{p}], \Pi_{rj}: j = 1 \dots rn]$. That is, if each firm sets the competitive price, they will have no incentive to deviate.

Theorem 1 (Existence). Let $\Upsilon > 0$. There exists r' such that for $r > r'$, $\underline{\theta}$ is an equilibrium.

Proof. Define $\sigma(\theta) - S(\theta) \equiv b$. $b > 0$ since $\Upsilon > 0$. Will firm i wish to deviate from $\underline{\theta}$ by raising his price $p_i > \theta$? Not if $d_{ri}(p_i, \theta) = 0$. We now show that there exists r' such that for $r > r'$, $d_{ri}(p_i, \theta) = 0$ for all $p_i > \theta$. If $(rn-1)\sigma(\theta) \geqslant r \cdot F(\theta)$, then $d_{ri}(p_i, \theta) = 0$, by P4 and P7 of A3. But we can ensure:

$$\begin{aligned} (rn-1)\sigma(\theta) &\geqslant rF(\theta) \\ (rn-1)b \quad &\geqslant rF(\theta) - rnS(\theta) + S(\theta) \\ &\geqslant S(\theta) \end{aligned}$$

This will be satisfied for $r > r'$ where

$$r' \geqslant \frac{S(\theta)}{n \cdot b} + \frac{1}{n} \tag{3}$$

Hence, when $r > r'$ firm i will not raise its price above θ. Standard arguments show that the firm will not cut its price. Therefore for $r > r'$, $\underline{\theta}$ is an equilibrium. QED.

At the competitive equilibrium $\underline{\theta}$, all the firms are producing at their profit maximizing outputs $S(\theta)$. However, at that price, firms are willing to supply up to b units more if it is demanded to avoid offending customers and incurring cost Υ. If firm i raises its price above θ, this will cause the $rn-1$ other firms still setting r either to expand their output to meet the additional demand if $(rn-1)\sigma(p) \geqslant rF(\theta)$, or to stay at $S(\theta)$ otherwise. In fact, it should be apparent that it may not need many firms to ensure that θ is an equilibrium. There are $rn-1$ firms each willing to produce b more units to meet demand: the extra demand generated as customers switch from i is only equal to $S(\theta)$ for any r. Equation (3) says that so long as the total number of firms rn is greater than $S(\theta)/b$ plus one, the extra demand will be satisfied. Again, recall that even quite small offence costs Υ can generate a significant value for b, since x_i is optimal at $S(\theta)$. We will illustrate Theorem 1 with $c(x) = cx^2/2$ and $F(p) = 1-p$ so that we have:

$$\theta = \frac{c}{n+c}; \quad S(\theta) = \frac{1}{n+c}; \quad b = \sqrt{(2\Upsilon/C)}$$

Hence the number of firms required to ensure that θ is an equilibrium is $r'n$ where:

$$r'n = 1 + \frac{\sqrt{c}}{(n+c)\sqrt{(2\Upsilon)}}$$

Clearly the number of firms $r'n$ need not be particularly large if c is small.

The knowledge that the competitive price can be in equilibrium is useful,

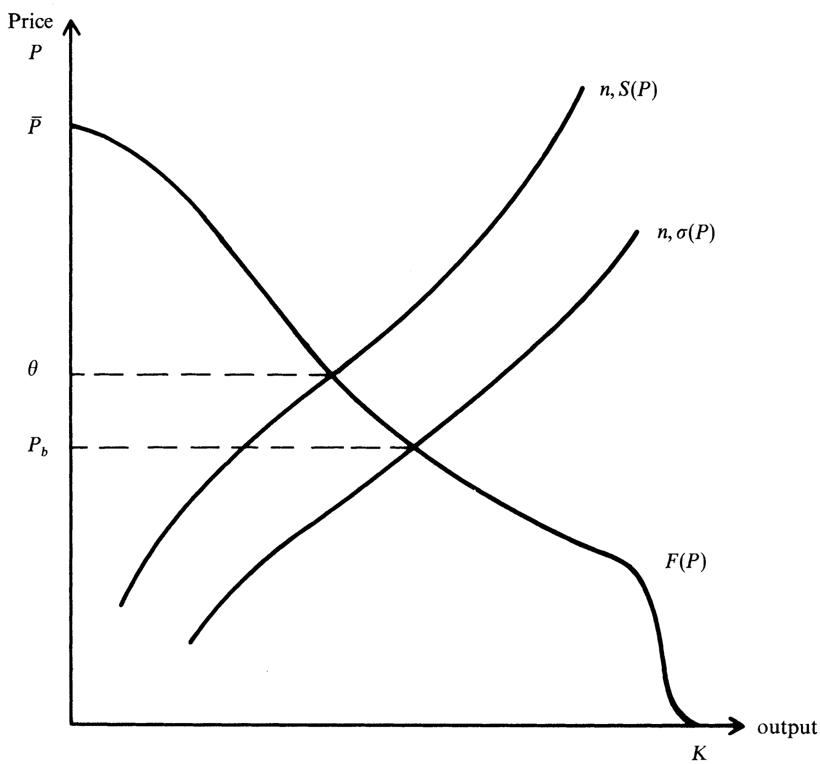

Figure 2
Definition of P_b

since the presence of offence costs Υ provides a plausible justification for the competitive market outcome. However, this may not be very useful unless we can characterize the full set of equilibria. This turns out to be quite easy to do. First, however, we need some more definitions.

Definitions. Let $\Upsilon \geqslant 0$.

(a) We define p_b by: $F(p_b) = n \cdot \sigma(p_b)$
(b) We define p_h by: $p_h \geqslant \theta$, and

(4) $$p_h \cdot F(p_h)/n - c(F(p_h)/n) = \xi(p_h) - \Upsilon$$

Note that under A1–A2, P_b exists (for Υ small) and is unique. P_b is the price at which demand equals the augmented supply $rn\sigma$, and is depicted in Figure 2. Clearly, $P_b < \theta$, and as Υ tends to zero, P_b tends to θ with $P_b = \theta$ when $\Upsilon = 0$. P_h is the price at which firms are indifferent between setting the same price (and sharing demand), and undercutting. This need not be uniquely defined

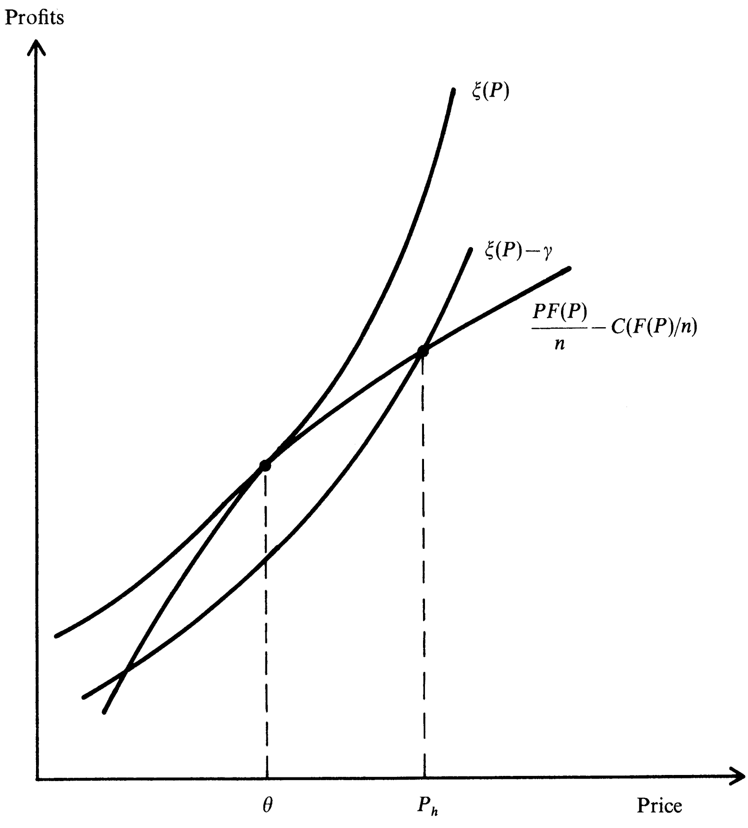

Figure 3
Definition of P_h

under A2, and for simplicity we will add the stronger assumption that demand is not too convex to make the LHS of (4) convex in P:

Assumption 4 (A4). $pF(p)/n - c(F(p)/n)$ is weakly concave in p on $[0, \bar{p}]$.

Under A1–A4, P_h is uniquely defined, as depicted in Figure 3. Again, note that at $\xi(\theta) = \theta F(\theta)/n - c(F(\theta)/n)$, since demand equals supply, so that $P_h > \theta$ for $\Upsilon > 0$, and $P_h \to \theta$ as $\Upsilon \to 0$, and $P_h = \theta$ for $\Upsilon = 0$. Armed with these definitions, we can now characterize the full set of equilibria in the Bertrand–Edgeworth model with customer offence. First we will show that there can be no multiple price equilibria, so long as Υ is not large. The reasoning is similar to that in standard Bertrand–Edgeworth models. However we state the argument fully in this slightly different context.

Proposition. Let $\Upsilon > 0$, with $\Upsilon < \xi(p_b)$. Then any pure-strategy Nash-equilibrium in $[[0, \bar{p}], \Pi_{rj}: j = 1 \dots rn]$ is a single price equilibrium.

Proof. Assume the contrary, so that two firms set different prices $p_i > p_j$. Either $d_{ri}(p_i, p_{-i}) = 0$ or $d_{ri}(p_i, p_{-i}) > 0$. If $d_{ri}(p_i, p_{-i}) = 0$, then $\Pi_{ri}(p_i, p_{-i}) = 0$: so firm i can set $p_i = p_b$ and earn at least $\xi(p_b) - \Upsilon > 0$, thus raising its profits (note that if it sets price p_b, then its demand is at least $\sigma(p_b)$ whatever prices other firms set). If $d_{ri}(p_i, p_{-i}) > 0$, then firm j is not meeting demand, and from A3 property 7 can raise its price a little, sell the same amount, thus raising its profits. QED.

We can now characterize the set of equilibria given Υ as r becomes large. Since these will all be single price equilibria from the Lemma, we will denote the rn vector where for all i $p_i = p$ by p.

Theorem 2. Let $\Upsilon > 0$.

(a) If $p \,\varepsilon (p_b, p_h]$, there exists r' such that for $r > r'$, p is an equilibrium.
(b) If $p \leqslant p_b$ or $p > p_h$, there exist r' such that for $r > r'$, p is not an equilibrium.

The proof is simple, but needs to exhaust several different cases, and so is in the appendix. Theorem 2 shows that as the industry becomes large the set of equilibria tends to $(p_b, p_h]$. As such, for any $\Upsilon > 0$ there will be prices above and below θ which will be also equilibria. The reason that prices above θ can be equilibria is that if firms undercut they will have to turn away customers and incur offence costs Υ: p_h is the price at which the firm is indifferent between setting the same price as the others, and ε-undercutting to capture industry demand with the consequence of turning away customers. Prices between θ and p_b can be equilibria because the potential augmented supply of firms exceeds industry demand at that price; hence, as at θ, if there are enough firms then any individual firm will face zero contingent demand if it raises its price. At p_b firms are on their augmented supply functions, being indifferent between serving all demand $x_i = \sigma(p_b)$, and turning away customers

$x_i = S(p_b)$: in both cases profits are $\xi(p_b) - \Upsilon$. This is a knife-edge situation, no matter how large the industry, if one firm raises its price, the demand for the remaining firms will exceed their augmented supply, so that from (2) they will cut supply to $S(p_b)$, and from A3 property 7 the firm raising its price a little will find that its demand has increased, enabling it to increase its profits.

From the definitions of p_h and p_b, we know that for small Υ any equilibria which do exist will be approximately competitive:

Corollary. Let $\varepsilon > 0$. There exists r' and Υ such that for $r > r'$, if p is an equilibrium, $|p - \theta| < \varepsilon$.

Proof. From Theorem 2(b), there exists r' such that, for $r > r'$ if p is in equilibrium then $p\ \varepsilon(p_b, p_h]$. But as $\Upsilon \to 0$, $p_h - p_b \to 0$, establishing the corollary. QED.

Note that the driving force behind the approximation result is the smallness of Υ: r does not need to be large to satisfy Theorem 2(b).

Lastly, we will discuss the generality and robustness of the results. Given the Bertrand–Edgeworth framework, A1–A3 are very general, and as we discussed we could have generalized A1–A2, but chose not to for expositional efficiency. The most crucial assumption is that there are *lump-sum* costs of offence. An alternative would be to allow the costs to vary with the level of unsatisfied demand. If we actually envisaged people queueing up at lowest priced firms first, and then if turned away going to higher priced firms, this would make sense. This, however, is a very mechanical interpretation of the contingent demand curve, and goes against the Bertrand–Edgeworth spirit of a frictionless market with perfectly informed customers. Contingent demand merely says how much a firm *could* sell, rather than the length of queues it will have. However, despite this, what would the effect be of allowing offence costs to vary with the number of unsatisfied customers? If there is a "lump sum" element, then of course there is no significant difference. However, if offence costs tend to zero as unsatisfied demand tends to zero, matters may be different.

Let us define e_i to be the level of unsatisfied demand (the 'number' or 'measure' of customers not served by firm i): $e_i = d_{ri} - x_i$. We now allow $\Upsilon_i = \Upsilon(e_i)$. We need to make some assumptions about $\Upsilon(\cdot)$, and a convenient additional assumption about the cost function:

Assumption 5 (A5).

(a) the cost function $c(\cdot)$ is twice continuously differentiable.

(b) $\Upsilon(\cdot)$ is continuously differentiable, non-decreasing, with $\Upsilon(0) = 0$, and as $e \to 0$, $\Upsilon(e)/e \to \Upsilon'(0) > 0$.

Theorem 3: A1–A3, A5. For r large enough, $\underline{\theta}$ is an equilibrium.

Again, the proof is lengthy, and so is in the appendix. What matters then, is not the lump-sum nature of Υ, but rather the *marginal cost* of unsatisfied customers at $e = 0$. Because there is no first-order cost to expanding output at $S(\theta)$, it is enough that there is a first-order cost to turning customers away. If loss of goodwill is to mean anything, then surely A5(b) must be satisfied. Intuitively, it would seem most natural that $\Upsilon(\cdot)$ is concave, with declining marginal customer offence costs, which eventually tend to zero. The notion of convexity is very unappealing, since it implies that offence costs may be unbounded, so that in a large industry, firms will be unwilling to undercut other firms for fear of attracting too many customers, which is implausible. Concavity, however, is consistent with Υ being bounded.

What can we say about other equilibria with variable offence costs? A formal analysis would be rather complex, needing much more structure for contingent demand than is provided by A3. However, we can note that so long as Υ is bounded, then the set of equilibria will be much as in theorem 2, with some prices above θ (for which undercutting is not worthwhile), and some below (since under A5b firms will expand output beyond the supply function to satisfy demand). So whilst the assumption of lump-sum offence costs is both conventional and convenient, the results of Theorems 1–2 do not rest on it.

IV. CONCLUSION

This paper provides a simple solution to the problem of non-existence of pure-strategy equilibria in Bertrand–Edgeworth models with strictly convex costs. The voluntary-trading constraint in standard Bertrand–Edgeworth models is generalized to allow for there being costs incurred when customers are turned away empty handed. So long as the industry is sufficiently large, the presence of such costs ensures that the competitive price will be an equilibrium. There will be other single price equilibria, but if the offence costs are small, all equilibria will be close to the competitive outcome. The main body of the paper assumes that there are lump-sum costs to turning customers away, which do not vary with the level of unsatisfied demand. However, it is not the lump-sum element that is crucial: rather it is that the marginal cost of turning away the first customer is positive.

The results of this paper will be useful, since they provide a simple justification for using the competitive equilibrium in a market with price-setting firms with strictly convex costs. As is well known, there is a paradox in standard competitive models: everyone is a 'price-taker', yet prices are presumed flexible. This paper suggests that we can avoid this problem because price-setting firms will behave as if they were price-takers. The assumptions made are very general, and represent a significant advance on standard Bertrand models with constant marginal cost. The model may also provide a theory of (limited) price rigidity in competitive markets. There is a continuum of possible equilibrium prices with the competitive price in the middle (as it were). Changes in demand (or supply shocks) could move this interval of prices around. For small changes, it is possible that the same price could remain in the equilibrium interval, so that no change in price need occur. Thus we would have a competitive market where prices only respond to shocks that are large enough. In a macroeconomic framework, even though small, such rigidities can have important implications for macroeconomic stability.

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APPENDIX

Proofs of Theorems 2 and 3.

Theorem 2. Let $\Upsilon > 0$.

(a) If $P \varepsilon (p_b, p_h]$, then there exists r' such that for $r > r'$ p is in equilibrium.

(b) There exists r', such that for $r > r'$, if $p \leqslant p_b$ or $p > p_h$, then p is not an equilibrium.

Proof. We already know that θ satisfies part (a) of the Theorem. We will break the proof into four cases: $p \leqslant p_b$ and $p > p_h$ (which together establish (b); $p_b > p > \theta$ and $\theta < p < p_h$ to establish (a).

To establish part (b) of the Theorem, consider:

Case 1. $p > p_h$. For $p > p_h$, then so long as $rF(p) \geqslant S(p)$:

$$\sup_{p_i < p} \Pi_i(p_i, p) = \xi(p) - \Upsilon$$

From the definition of p_h, noting A4, it will pay firms to undercut p by some small amount, despite incurring offence costs Υ.

Case 2. $p \leqslant p_b$. Here for all firms i, $d_{ri}(p) \geqslant \sigma(p)$, with equality for $p = p_b$, strict inequality $p < p_b$. For $p = p_b$, firms produce $\sigma(p_b)$, earning $\xi(p_b) - \Upsilon$; for $p < p_b$, $S(p)$, earning $\xi(p) - \Upsilon$. If firm i raises its price, it can increase profits from property 7 of A3.

$$\begin{aligned}\lim_{\varepsilon \to 0} d_{ri}(p+\varepsilon, p) &= rF(p) - (rn-1)S(p) \\ &> S(p)\end{aligned}$$

This holds true at $p = p_b$ because when i raises price, other firms j switch from $x_j = \sigma(p_j)$ to $S(p_j)$ since $d_{rj}(p_i, p) > \sigma(p_j)$ when $p_i > p$. Hence, from property 6 there exists $p' \varepsilon (p, \bar{p})$ such that $d_{ri}(p', p) = S(p')$. So firm i will increase its profits to $\xi(p') > \xi(p) - \Upsilon$. Hence, $p \leqslant p_b$ is not an equilibrium. This establishes Part (b) of the Theorem.

Turning to Part (a), we will consider cases 3–4:

Case 3. $p_b < p < \theta$. For prices in this interval:

$$nS(p) < F(p) < n\sigma(p)$$

and each firm produces $x_i = F(p)/n$. No firm will want to undercut (and incur offence costs) so long as $rF(p) > \sigma(p)$. We will show that for r large enough, no firm i will want to raise its price, since: $d_{ri}(p_i, p) = 0$ for $p_i > p$. If firm i raises its price, the residual demand is zero when:

$$(5) \qquad rF(p) - (rn-1)\sigma(p) \leqslant 0$$

For $r > \dfrac{-\sigma}{F(p) - n\sigma(p)}$ inequality (5) is satisfied (i.e. there is excess supply) so that from Property 7, d_{ri} is zero for $p_i > p$, since demand is met at the lower price.

Case 4. $\theta < p < p_h$. For prices in this range, firms are tempted to undercut. So long as $rF(p) > \sigma(p)$,

$$\sup_{p_i < p} \Pi_{ri}(p_i, p) = \xi(p) - \Upsilon$$

By definition, under A4, it will not pay to undercut (see Figure 3).

This establishes Part (b) of the Theorem. QED.

Proof of Theorem 3.

As a preliminary, we need to consider the cost to the firm ($\Delta\Pi$) of expanding its output by Δx units of output above the optimal output $S(\theta)$. By the Mean Value Theorem, given Δx there exists $x' \varepsilon [S(\theta), S(\theta) + \Delta x]$ s.t.:

$$\Delta\Pi = -\left(\theta - \frac{\partial c(x')}{\partial x}\right)\Delta x$$

Similarly, there exists $x'' \varepsilon [S(\theta), x']$ such that:

$$\frac{\partial c(x')}{\partial x} = \frac{\partial c(S(\theta))}{\partial x} + \frac{\partial^2 c(x'')}{\partial x^2} \cdot (x' - S(\theta))$$

Hence:

$$\Delta\Pi = \frac{\partial^2 c(x'')}{\partial x^2} \cdot (x' - S(\theta)) \cdot \Delta x$$

Under A5a, $\partial^2 c/\partial x^2$ is bounded from above by $c > 0$ (since $x \varepsilon [0, \sigma(\bar{p})]$), so that the loss in profits obeys:

$$(6) \qquad \Delta\Pi \leqslant c \cdot \Delta x^2$$

since $(x' - S) < x$. This provides an upper bound for the loss the firm makes by expanding output beyond the profit maximizing level.

To establish Theorem 3, we show that if the industry is large enough, then when a Nash-deviant raises its price, to $p_i > \theta$, then the $rn - 1$ remaining firms will raise output to cover the extra demand of $S(\theta)$, thus driving i's profits to 0. Hence θ will be an equilibrium if expanding output by $\Delta x = S(\theta)/(rn-1)$ leads to less losses than incurring offence costs $\Upsilon(S(\theta)/rn - 1)$. From (6) above this means that:

$$(7) \qquad \Upsilon\left(\frac{S(\theta)}{rn-1}\right) \geqslant c \cdot \left(\frac{S(\theta)}{rn-1}\right)^2$$

Under A5b, by the mean value theorem:

$$(8) \qquad \Upsilon\left(\frac{S(\theta)}{rn-1}\right) = \Upsilon(0) + \Upsilon'(e')\left(\frac{S(\theta))}{rn-1}\right)$$

where $0 \leqslant e' \leqslant S(\theta))/(rn-1)$. Since Υ is continuously differentiable, and under A5b $\Upsilon'(0) > 0$, for rn large enough, there exists $\bar{\Upsilon} > \Upsilon'(0)/2 > 0$ s.t.:

$$(9) \qquad \min_{e\varepsilon[0, S(\theta)/(rn-1)]} \Upsilon'(e) > \bar{\Upsilon}$$

Since $\Upsilon(0) = 0$, from (8):

$$(10) \qquad \Upsilon\left(\frac{S(\theta)}{rn-1}\right) \geqslant \bar{\Upsilon}\cdot\left(\frac{S(\theta)}{rn-1}\right)$$

Returning to our equilibrium condition (7), using (10) we have:

$$\bar{\Upsilon}\cdot\frac{S(\theta)}{rn-1} \geqslant c\cdot\left(\frac{S(\theta)}{rn-1}\right)^2$$

$$\bar{\Upsilon} \geqslant c\cdot\frac{S(\theta)}{rn-1}$$

But this equilibrium condition must surely be satisfied for rn large enough, i.e. $rn > r'n = 1 + c[S(\theta)/\bar{\Upsilon}]$. Hence for $r > r'$, and r large enough to satisfy (9), any firm setting $p_i > \theta$ will sell nothing, $d_{ri} = 0$, thus establishing the Theorem. QED.

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