

Approximate Bertrand Equilibria in a Replicated Industry

HUW DIXON
Birkbeck College

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This paper considers the existence and properties of approximate Bertrand equilibria in a replicated industry. Price setting firms produce a homogeneous product with weakly convex costs. The main results are that: (a) Given $\varepsilon > 0$, an ε -equilibrium exists if the industry is large enough; (b) If the ε is small enough, and the industry large enough, any ε -equilibrium is approximately competitive. These results depend on how contingent demand is specified.

During a price war between two petrol stations in Winnipeg, Mr Hafy Carnet reduced his gasoline price from 50 cents to 10 cents a litre, whereupon Mrs Sharon Willard, his neighbour, cut her gasoline to 1.6 cents a litre. Police were called when, having lost three hundred customers, Mr Carnet, "who completely forgot the rules of the market", announced through a loud-hailer that he would pay 3 cents to anyone who filled their tank at his pumps.

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Price-setting is the institutional form of pricing in many sectors of industrialised economies. This paper develops a framework which generalises the models of price-competition associated with Bertrand (1883) and Edgeworth (1897). Essentially such models describe a trading process in an industry producing a homogeneous product where firms set prices which perfectly informed households treat parametrically (see Allen and Hellwig (1982, 1983), d'Aspremont and Gabszewicz (1980), Brock and Scheinkman (1985), Dasgupta and Maskin (1986), Dixon (1984a), Kreps and Scheinkman (1983), Levitan and Shubik (1972), Osborne and Pitchik (1985), Shapley (1957), and Shubik (1955, 1959)). These papers have retained one of Edgeworth's basic assumptions, namely that firms have constant average costs up to capacity (exceptions being Shapley (1957), Shubik (1958), Dixon (1984a)). The main purpose of this paper is to show that this simple Bertrand-Edgeworth framework can be considerably generalised.

The model presented in this paper has two main features of interest. Firstly, we make a very general assumption about firms' cost functions, namely that firms have continuous and (weakly) convex total costs. This assumption embraces the Bertrand case (constant average costs), the Edgeworthian case, and the more orthodox case of strictly convex costs. Secondly, we assume that there are strictly positive lump-sum costs of decision and price-adjustment. The presence of such costs is certainly very plausible. It also solves a major conceptual difficulty within such price-setting models—the problems associated with the non-existence of pure strategy equilibria and the recourse to less plausible mixed-strategy solutions.

This paper does not model costs of decision and adjustment explicitly. Rather, we employ the solution concept of an epsilon-equilibrium or "approximate equilibrium". An ε -equilibrium occurs when each agent is within ε of his best payoff given the actions

of the other agents (we shall be restricting ourselves to the case of Nash ε -equilibria). We employ the ε -equilibrium concept because it gives our results a greater generality than if we explicitly modelled costs of decision and adjustment. Whilst such costs are the most natural interpretation of the ε , we could also interpret it as relevant due to bounded rationality in a more general sense.

The results in this paper concern the existence of approximate Bertrand equilibria, and their approximation to the competitive equilibrium when we replicate the industry. The starting point of our analysis is Theorem 1 (non-existence), which shows that with strictly convex costs, the standard assumptions that demand and cost functions possess bounded derivatives imply that no strict equilibrium will exist.¹ Turning to approximate equilibria, Theorem 2 (existence) shows that given $\varepsilon > 0$ (no matter how small), if the industry is large enough, then an ε -equilibrium will exist. Theorem 3 (approximation) demonstrates that if ε is small enough, and the industry large enough, then any ε -equilibrium will be approximately competitive.

It needs to be made clear at the outset that the Theorem 2 (existence) holds only for one specification of contingent demand, "CCD" defined in Assumption A3 below, which originates in Levitan and Shubik (1972). In the literature there are two types of contingent demand functions used, which give the demand for a higher priced firm when demand has been partly satisfied at lower prices. Under CCD the higher priced firm faces the industry demand less the quantity supplied by lower priced firms. Under the alternative FCFS (see Assumption A4) contingent demand is the proportion of industry demand left unserved by lower priced firms (this originates in Edgeworth (1897)). The difference between these specifications can be seen as arising from different rationing mechanisms that operate when there is excess demand for a lower priced seller (see Dixon (1986)). Under CCD all households receive a proportion of what they want: under FCFS a proportion of households receive all they want, and the rest nothing. The two specifications of demand have very different properties.² Theorem 2 works for CCD because as the industry is replicated, the firm's contingent demand becomes infinitely elastic at the competitive price, and hence the incentive for any one firm to deviate from the competitive price tends to zero. Under FCFS, however, replication does not affect the contingent demand at the competitive price (Proposition 1), so that even in a very large industry the firm will have the same incentive to deviate from the competitive price as in a small industry.

We see three main reasons why these results are important. First, they generalise the Bertrand-Edgeworth framework, indicating that it can be applied to cases where firms have non-trivial cost functions. Secondly, the model presents a possible account of how a competitive equilibrium might occur. If we recall Arrow's Paradox, in a competitive market all agents are assumed "price-takers", yet the price is assumed to be "flexible" and attain the equilibrium value (Arrow (1959)). The model presented in this paper gives an account of how price-setting firms might come to behave as if they were price-takers (see Dixon (1982)). In a large industry with small costs of decision and price adjustment, we can use the Theorems of existence and approximation to give an account of how the competitive equilibrium can come about. Thirdly, price dispersion is the typical case in an approximate Bertrand equilibrium. Even in many markets which would commonly be accepted as "competitive", there is rarely only one price charged. The presence of lump sum costs of decision and price adjustment thus provides a simple and intuitively satisfying account of such "limited" price dispersion. This complements the standard accounts of price dispersion due to consumer search costs (as in Salop and Stiglitz (1977), Sadanand and Wilde (1982) *inter alia*).

The closest results to this paper concern mixed-strategy equilibria in the Bertrand-Edgeworth framework. Shubik argued for the general Bertrand-Edgeworth case with mixed-strategies that as the number of firms tended to infinity, the expected price each firm sets will tend to the competitive price (1959, p. 123, 1955). This result has been further investigated by Allen and Hellwig (1986). Our result is rather different from this, since the Approximation Theorem can apply to small industries, although it is only in large industries that we can be confident that an ε -equilibrium exists when ε is small. The most important difference is that in this paper, "approximation" means that each firm sets its price close to the competitive price. With mixed-strategies only the expected price converges as the industry is replicated: Allen-Hellwig (1986) show that even in large industries firms will set prices that are not close to the competitive price with a positive probability. Our results also differ from the convergence of Cournot-Nash equilibria to the competitive equilibrium (see Novshek and Sonnenschein (1983), Roberts (1980) *inter alia*). This literature considers the approximation of *strict* Cournot equilibria to competitive outcomes in general equilibrium economies. In the Cournot-Nash framework, of course, the Law of one price is imposed via the inverse demand correspondence.

1. THE GENERAL BERTRAND-EDGEWORTH FRAMEWORK

Without replication there is a set g of n firms producing a homogenous product. If the industry is replicated r times, each firm in g is replicated r times, the resulting set being denoted g_r . Each firm $i \in g_r$ is free to set its own price $p_i \in [0, \infty)$, the m vector of prices being denoted \underline{p} .

Assumption A1. Costs. Each firm $i \in g$ has a total cost function $c_i: R_+ \rightarrow R_+$, which is strictly increasing, continuous and (weakly) convex in output x_i .

From the cost function we derive the firm's supply-correspondence $s_i(p_i)$, and corresponding "supply functions" S_i and σ_i which are derived by taking the supremum and infimum of the correspondence:

Definition 1. (a) Supply-correspondence $s_i: R_+ \rightarrow R_+^*$.

$$s_i(p) = \operatorname{argmax}_{x_i \in [0, \infty]} p \cdot x_i - c_i(x_i)$$

(b) Supply functions:

$$S_i(p) = \sup \{s: s \in s_i(p)\}$$

$$\sigma_i(p) = \min \{s: s \in s_i(p)\}.$$

We do not impose an upper bound on x_i because this would be inappropriate in the context of replication, and include ∞ as a possible value for x_i (by convention $c_i(\infty) = \operatorname{Lim} c_i$ as $x_i \rightarrow \infty$). From Definition 1(a) note that s_i is a closed, convex valued, upper-hemicontinuous correspondence, and s_i is monotonic (let $q > p$, then for any $s^q \in s_i(q)$, $s^p \in s_i(p)$, $s^q \geq s^p$). Hence s_i is multivalued only at a countable number of points, as depicted in Figure 1. Given the properties of s_i , both σ_i and S_i are non-decreasing, continuous almost everywhere, whilst S_i is right-continuous and left-upper-semi-continuous, and σ_i left-continuous and right-lower-semi-continuous.³ Also $\sigma(p) \geq S(q)$ whenever $p > q$. We make the assumption that firms have "lexicographic preferences" in the sense that they prefer to produce the largest of any outputs yielding the same profit.

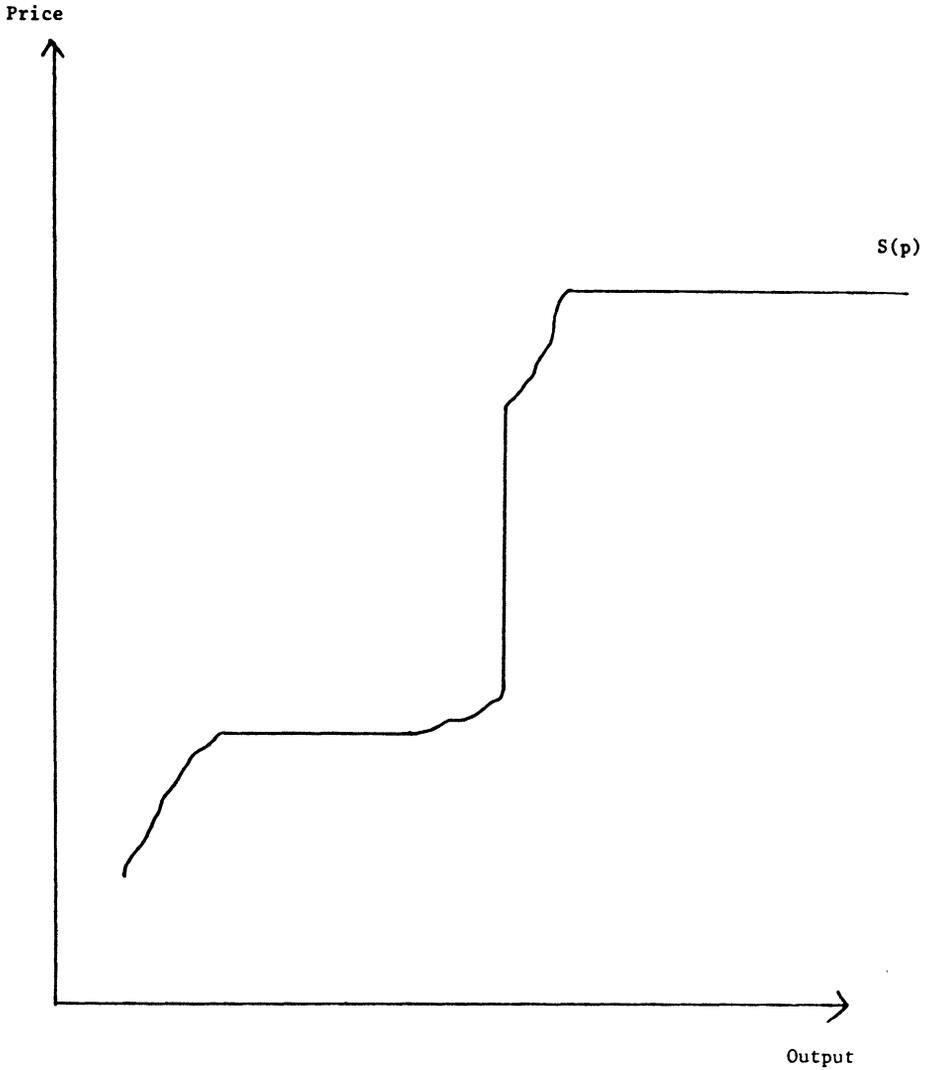


FIGURE 1
Supply correspondence

This is a reasonable assumption, and one that is implicit in the standard treatment of the Bertrand model with constant costs,⁴ although it is not necessary for our results.⁵ Under lexicographic preferences the firm's *desired* trade is therefore given by the supply function $S_i(p)$.

The industry demand function gives the demand when all firms set the same price:

Assumption A2. Industry demand. $F: R_+ \rightarrow [0, \Xi]$. Without replication, industry demand is given by $F(p)$, where:

- (a) $F(0) = \Xi$, where Ξ is positive and finite
- (b) there exists $p^* \in [0, \infty)$ such that: $p^* = \sup \{p \in R_+: F(p) > 0\}$
- (c) F is continuous, non-increasing, and strictly decreasing when strictly positive.

The assumption of continuity (c) can be relaxed: all that is required for the results in this paper is that industry demand is left-continuous. This paper includes little explicit analysis of demand, but it can be pictured as arising from a continuum of perfectly informed non-strategic consumers.

Assumptions 1-2 are together very general, and embrace the usual special cases dealt with in the literature. These include the standard Bertrand case where there is constant average costs for any positive output, and the Edgeworthian case of constant costs up to capacity.⁶

If the industry is replicated r times, then the industry demand becomes $r \cdot F(p)$ and the set of firms g_r . The competitive price θ is unaffected by replication, and is defined:

$$\theta =_{\text{def}} \inf \{ p: F(p) - \sum_{i \in g} S_i(p) < 0 \} \tag{1}$$

which is illustrated in Figure 2. To avoid trivialities, we assume that for all firms $S_i(\theta) > 0$,

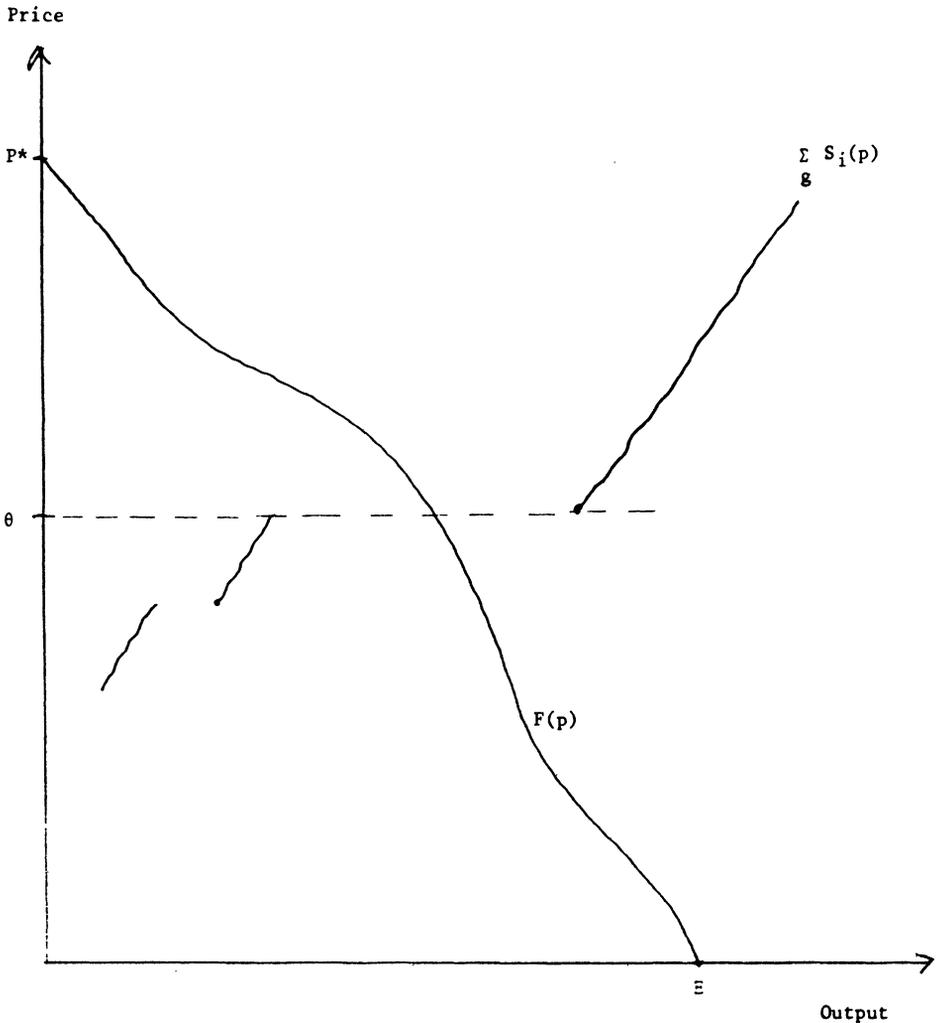


FIGURE 2
The competitive price

and that $p^* > \theta > 0$. It is important to note that from (1) and Definition 1(b):

$$F(\theta) - \sum_{i \in g} S_i(\theta) \leq 0 \leq F(\theta) - \sum_{i \in g} \sigma_i(\theta). \quad (2)$$

In order to model price-setting behaviour, we need to know the demand facing firm j given the prices and quantities offered by other firms, the firm's contingent demand function. In general contingent demand depends on prices set and quantities offered by firms. However, since we assume firms have lexicographic preferences the quantity offered becomes a function of price—if firm j has set price p_j , he will offer to sell up to $S_j(p_j)$. Contingent demand for j in an r -replica industry can then be written as a function of prices only:

Definition 2. Contingent Demand for Firm j , $d_{nj}: R_+^m \rightarrow R_+$.

$$d_{nj} = d_{nj}(\underline{p}).$$

We have analysed the possible specifications of contingent demand in some detail elsewhere (Dixon (1986)). We shall employ the two standard specifications found in the literature. They both assume perfectly informed consumers who buy from the lowest priced firm that will sell to them. They differ over how the demand for a higher priced firm is determined when households have already purchased from a lower priced firm.

The first, *First-come-first-served* (FCFS), originates in Edgeworth (1897), and has been used more recently in Allen and Hellwig (1983, 1984), Bekman (1965), and Dasgupta and Maskin (1986). The most natural interpretation of this specification is that if there is excess demand for the output of a lower priced firm(s), then a subset of consumers are sold as much as they want, and hence have no residual demand for higher priced firms (we ignore any marginal customer who is only partly satisfied). The demand for higher-priced firms is given by the sum of the demand functions of the customers unsatisfied at lower prices. If consumers are heterogeneous, then the contingent demand under FCFS will be random, depending on who has been served first. We assume that demand is non-random, which can be justified by the assumption that household preferences are identical and homothetic (Dixon 1986).

The second specification, *Compensated Contingent Demand* (CCD), originates in Levitan and Shubik (1972), and has been used by Brock and Scheinkman (1985), Kreps and Scheinkman (1983), Osborne and Pitchik (1985). Under CCD the demand for the higher priced firms is the industry demand minus the total quantity sold by lower priced firms. This is what the contingent demand would be if there was only one household in the market, and any "income effects" from lower priced purchases were compensated. With many identical households it can be seen as arising from an "equal shares" rationing rule, which means that if there is excess demand for a firm(s), then each household receives an equal share of the available output. This can be generalized to heterogeneous households if there is a proportional-rationing scheme (as employed by Dubey (1982))—households receive a certain proportion of their demand, this being determined so as to equate purchases with supply at that price. Of course, proportional rationing is manipulable, since any household will receive more if it asks for more.⁷ Alternatively, households with unit demand functions and "reservation price rationing" (households with higher reservation prices are served first) give rise to CCD (Gelman and Salop (1983)).

With excess supply the exact method by which demand is allocated amongst firms setting the same price is not important for our results. The existing literature tends to make very specific assumptions about this.⁸ However, since households are indifferent between firms setting the same price, there is no obvious general reason for a specific

rule (Dixon (1986)). We simply require the contingent demand for any firm j to lie between an upper bound V_j (based on the optimistic assumption that households will shop at j first), and a lower bound W_j (based on the pessimistic assumption that households shop at j last). If no other firms are setting the same price as j , then $V_j = W_j$. The only conditions that we impose on contingent demand functions d_j are that (a) contingent demand for j lies between W_j and V_j ; (b) the contingent demands of firms setting the same price “add up” so that only one side of the market is rationed and there is voluntary trading;⁹ (c) demand for one firm is positive iff it is positive for all firms setting the same price.

Assumption A3. Compensated Contingent Demand CCD. Let $i, j \in g_r$.

Define

$$W_{rj}(\underline{p}) = \max [0, r \cdot F(p_j) - \sum_{p_i \equiv p_j}^{i \neq j} S_i(p_i)]$$

$$V_{rj}(\underline{p}) = \max [0, r \cdot F(p_j) - \sum_{p_i < p_j} S_i(p_i)]$$

(a) $d_{rj}(\underline{p}) \in [W_{rj}(\underline{p}), V_{rj}(\underline{p})]$

(b) Adding up: $\sum_{p_i = p_j}^{i=j} \min [S_i(p_i), d_{ri}(\underline{p})] = \min [V_{rj}(\underline{p}), \sum_{p_i = p_j}^{i=j} S_i(p_i)]$

(c) If $p_i = p_j$, $d_{ri} > 0$ iff $d_{rj} > 0$.

In order to specify contingent demand under FCFS, first define:

$$k_{ri}(p_i) = S_i(p_i) / r \cdot F(p_i)$$

$$K_{rj}(\underline{p}) = \max [0, 1 - \sum_{p_i < p_j} k_{ri}(p_i)]$$

k_{ri} represents the proportion of the total industry demand at p_i that firm i can satisfy. K_{rj} gives the proportion of consumers left over once they have tried to purchase from lower priced firms.

Assumption A4. First Come First Serve. Let $i, j \in g_r$, $i \neq j$.

Define

$$W_{rj}(\underline{p}) = \max [0, K_{rj}(\underline{p}) \cdot r \cdot F(p_j) - \sum_{p_i = p_j}^{i \neq j} S_j(p_i)]$$

$$V_{rj}(\underline{p}) = K_{rj}(\underline{p}) \cdot r \cdot F(p_j).$$

(a) $d_{rj}(\underline{p}) \in [W_{rj}(\underline{p}), V_{rj}(\underline{p})]$

(b) Adding up: $\sum_{p_i = p_j}^{i=j} \min [S_i, d_{ri}] = \min [V_{rj}(\underline{p}), \sum_{p_i = p_j}^{i=j} S_i(p_i)]$

(c) As in Assumption A3(c)

Under both specifications of contingent demand, d_{rj} will be left lower semi-continuous in own price p_j . With either specification, if the output of firms setting some price, p say, is positive, then the outputs of all firms setting prices below p are given by $S_i(p_i)$.

Having defined the firm's demand and outputs as a function of prices set, we can now define the payoffs as functions of prices set. Profits are of course $p_j \cdot x_j - c_j(x_j)$. The profit function $\zeta_j: R_+ \rightarrow [0, \infty]$ we define as:

$$\zeta_j(p_j) \stackrel{\text{def}}{=} p_j \cdot s_j(p_j) - c_j(s_j(p_j)) \tag{3}$$

The payoff function for the model is then $\pi_{rj}: R_+^m \rightarrow R_+$

$$\pi_{rj}(\underline{p}) = \begin{cases} \zeta_j(p_j) & \text{if } d_{rj}(\underline{p}) \geq \sigma_j(p) \\ p_j \cdot d_{rj}(\underline{p}) - c_j(d_{rj}(\underline{p})) & \text{otherwise} \end{cases} \tag{4}$$

Thus we have a game where firms' payoffs depend only on prices, which inverts the logic of the Cournot-Nash model, where the inverse industry demand curve makes the market-price a function of quantities, and hence profits depend only on quantities. We now define:

*Definition 3. ε -equilibrium.*¹⁰ Let $\varepsilon \geq 0$. If $\underline{p} \in R_+^m$ is an ε -equilibrium, there exists no firm $j \in g$, and no price $q \in [0, \infty)$ such that:

$$\pi_{rj}(q, p_{-j}) - \pi_{rj}(\underline{p}) > \varepsilon$$

or equivalently:

$$\sup \pi_{rj}(q, p_{-j}) - \pi_{rj}(\underline{p}) \leq \varepsilon$$

In an ε -equilibrium, should one exist, we have an m vector of prices set by firms. For large ε any price-configuration may be an equilibrium. The smaller ε becomes, the more price-configurations are ruled out. We are not assuming that firms are required to maintain fixed prices over the trading period, as in Benassy (1976) and Iwai (1982, pp. 14-15). Firms are free to change prices whenever they wish to do so. Given the lump sum costs of decision and adjustment, however, it is optimal ex-post for firms to keep their prices fixed in equilibrium. Thus in the model presented here both prices and quantities are flexible. There is of course an asymmetry between prices and quantities in this model, since the costs of varying output are continuous and convex by Assumption 1, whilst the cost of changing price are discontinuous and non-convex.

2. EPSILON-EQUILIBRIA IN A REPLICATED INDUSTRY

Before considering the general existence of ε -equilibria, we shall first consider the problem of non-existence of strict equilibria.

Theorem 1 (Non-existence). *Assume Assumption A1-2 and either Assumption A3 or A4. Let $r \in z_+$, and $m > 1$. If F is differentiable, F' is bounded from below and c_j are strictly convex and differentiable, then no strict equilibrium exists.*

Proof. Note that if c_j are strictly convex then s_j are single valued and bounded functions for p_j on $[0, p^*]$. As is well known, the only strict pure-strategy equilibrium possible is $\underline{\theta}$ where all firms set the competitive price (Shubik (1959, p. 100 Theorem 2)).

To see that $\underline{\theta}$ is not an equilibrium, consider the contingent demand for a firm raising his price to q_j . Under CCD we have:

$$\pi_{rj}(q_j, \theta_{-j}) = q_j \cdot [r \cdot F(q_j) - \sum_{i \neq j} s_i(\theta)] - c_j(r \cdot F(q_j) - \sum_{i \neq j} s_j(\theta))$$

Hence:

$$\partial \pi_{rj} / \partial q_j = (q_j - c'_j) \cdot r \cdot F' + s_j(\theta).$$

Evaluated at $q_j = \theta$, given that F' is bounded, $\partial \pi_{rj} / \partial q_j = s_j(\theta) > 0$. This is, of course, exactly analogous to Hotelling's Lemma, and holds for much the same reason. Since the output $s_j(\theta)$ is optimal for $q_j = \theta$, and $\theta = c'$, a small increase in q_j from θ has no first-order effect on profits. Since c_j is strictly convex, c'_j is also bounded over some suitably chosen closed interval, $[0, s_i(p^*)]$ say. The argument also holds for FCFS. ||

Theorem 1 suggests that existence will be due either to some kinkiness or weak convexity in the cost function, or a demand curve which is horizontal around θ .¹¹

We shall now consider the general properties of ε -equilibria under replication when $\varepsilon > 0$. The first result is that for any $\varepsilon > 0$, no matter how small, under CCD an ε -equilibrium will exist if the economy is sufficiently large.

Theorem 2 ((Existence). *Under Assumption A1–A3 (CCD), let*

$$\varepsilon_r =_{\text{def}} \inf \{ \varepsilon : \text{an } \varepsilon\text{-equilibrium exists in an } r\text{-replica industry} \}$$

For large r ε_r is well defined, and as r tends to ∞ , ε_r tends to 0.

Proof. Let $\varepsilon > 0$. Define q_{rj} as the price above which firm j 's contingent demand is zero, given that the other firms ($i \neq j, i \in g_r$) set the competitive price:

$$q_{rj} = \inf \{ q : d_{rj}(q, \theta_{-j}) = 0 \}$$

We now show that under CCD, $q_{rj} \rightarrow \theta$ as $r \rightarrow \infty$. For $r \geq 2$, either:

(a) $r \cdot F(\theta) - \sum_{i \neq j} S_i(\theta) \leq 0$

or

(b) $r \cdot F(\theta) - \sum_{i \neq j} S_i(\theta) \geq 0$.

If (a) holds (as in the Bertrand case), then $q_{rj} = \theta$. If (b) holds then $q_{rj} > \theta$. Let $\lambda > 0$. There exists r_0 such that for $r > r_0$, $q_{rj} < \theta + \lambda$. To see why we can expand the sum of other firms desired outputs at θ :

$$r \cdot F(q_{rj}) = r \cdot \sum_{i \in g} S_i(\theta) - S_j(\theta) \quad (i = j \text{ if applicable})$$

so that

$$F(q_{rj}) - \sum_{i \in g} S_i(\theta) = \frac{-1}{r} \cdot S_j(\theta). \tag{5}$$

But from (1), $\sum_{i \in g} S_i(\theta) \geq F(\theta)$, so that from (5):

$$F(q_{rj}) - F(\theta) \geq -1 \cdot S_j(\theta)/r. \tag{6}$$

Since F is continuous and strictly decreasing when positive, we can choose r large enough so that the R.H.S. of (6) is small enough to ensure that $q_{rj} < \theta + \lambda$.

Given that q_{rj} tends to θ , the firms incentive to defect from θ will also tend to 0. Either $S_j(\theta)$ is bounded from above by some $\Lambda > 0$, or it is not. If $S_j(\theta)$ is unbounded, then $q_{rj} = \theta$ for $r \geq 2$. Hence:

$$\sup_{q \geq 0} \pi_{rj}(q, \theta_{-j}) - \pi_{rj}(\theta) = 0. \tag{7}$$

If $S_j(\theta)$ is bounded, then:

$$\sup_{q \geq 0} \pi_{rj}(q, \theta_{-j}) - \pi_{rj}(\theta) \leq (q_{rj} - \theta) \cdot \Lambda. \tag{8}$$

Hence as $r \rightarrow \infty$ and $q_{rj} \rightarrow \theta$, from (7) and (8):

$$\sup [\pi_{rj}(q, \theta_{-j}) - \pi_{rj}(\theta)] \rightarrow 0. \quad \parallel$$

The intuition behind Theorem 2 is that as the industry is replicated, whilst the industry elasticity of demand remains constant, the firm's elasticity tends to infinity around θ . Under CCD the slope of the contingent demand function for $p_j > \theta$ (when positive) becomes r times the slope of F , whilst the size of the firm remains constant. This is depicted in Figure 3 for the case where $S_j(\theta)$ is bounded. If we interpret CCD as arising

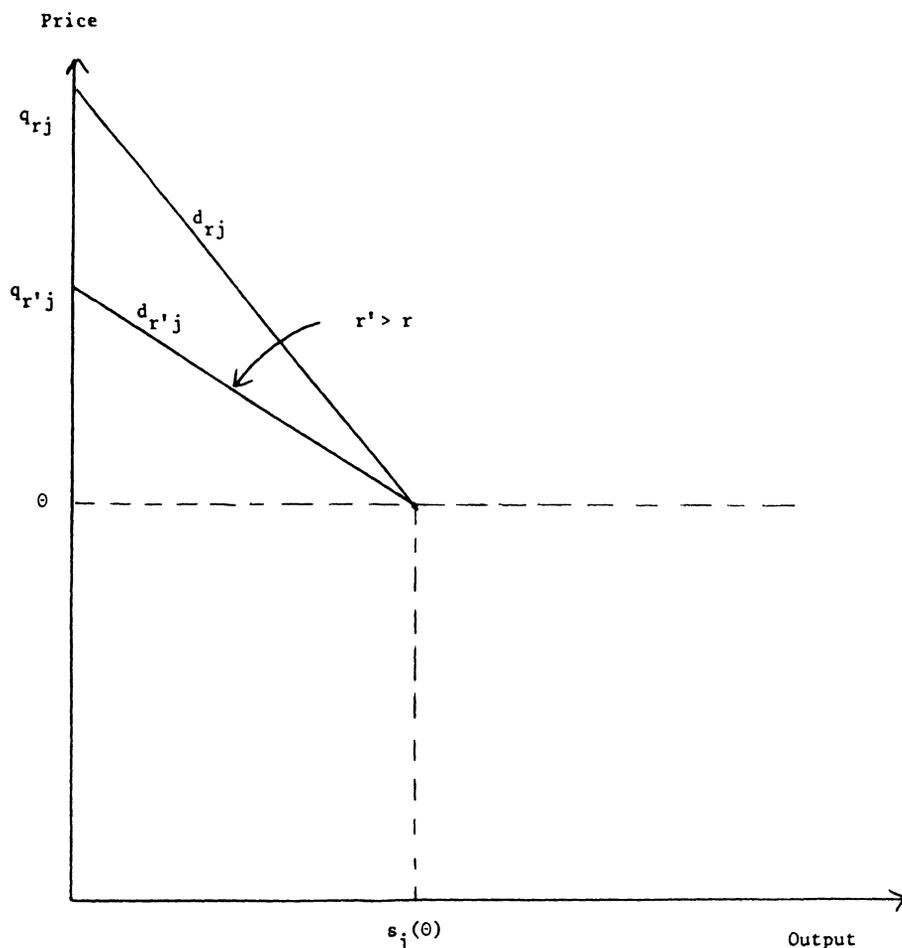


FIGURE 3

Contingent demand around θ as the industry is replicated

from an equal-shares or proportional-rationing scheme, the result is also intuitive. In a large industry starting from θ , when firm j raises his price the reduction in total supply at θ is relatively small. Under equal shares or Proportional rationing, each individual household will only be affected slightly by this reduction, since it is spread over all households. By continuity, the additional amount any householder will be willing to pay to satisfy his frustrated demand will be very small. Hence the contingent demand facing firm j will be almost perfectly elastic.

Theorem 2 is very specific in the sense that the proof only deals with the competitive price. Theorem 3 embraces the set of all approximate equilibria: if ε is small enough, and the industry large enough, then in any ε -equilibrium in which all firms have positive output, all prices set will be close to the competitive price. To see why we need to impose the restriction that all firms have positive outputs, consider the standard Bertrand model where all firms have the same constant costs with no capacity limit. If there are more than two firms, then *any* price vector is a strict equilibrium so long as two firms set the competitive price. Since all firms earn zero profits whatever price is charged, even for $\varepsilon = 0$ there exist equilibria with firms setting prices arbitrarily far from θ with zero output.

Theorem 3 (Approximation). *Assumption A1-2, A3 or A4. Let $\lambda > 0$. There exists $\varepsilon > 0$ $r_0 > 0$, such that if $\underline{p} \in R_+^m$ is an ε -equilibrium, and $\underline{x} \gg 0$, then*

$$|p_j - \theta| < \lambda \quad \text{for all } j \in g_r, r > r_0.$$

Proof. See appendix. \parallel

Taken together, the existence and approximation Theorems 2 and 3 are very general. They imply that in a “large” industry with small costs of decision and price adjustment, the trading process represented by Assumptions A1-4 will give rise to an outcome that is almost competitive. The approximation will consist both in the fact that the prices set are close to θ , and that industry output is close to its competitive level. Approximation Theorem 3 also implies that any price dispersion will be small. How large is “large” in this context is not theoretically determinable, depending on such factors as the elasticity of industry demand and the firm’s cost functions. In the Bertrand case of identical firms with constant returns to scale a strict equilibrium exists with two firms, and both firms set θ .

The Existence Theorem 2 holds only for CCD, not for FCFS. For FCFS, the elasticity of demand may be unaffected by replication:

Proposition 1. *Assumptions A1-2, A4. If all firms have strictly convex costs, there exist constants $M_j \geq 0$ such that for all $r, j \in g_r$,*

$$\sup \pi_{rj}(q, \theta_{-j}) - \zeta_j(\theta) = M_j.$$

Proof. Given that other firms set θ , consider j ’s contingent demand if it sets price $q \in (\theta, p^*]$. If all firms have strictly convex costs, then s_j is single valued, and from Definition 3, $F(\theta) = \sum_g s_i(\theta)$.

$$\begin{aligned} d_{rj}(q, \theta_{-j}) &= \max \left[0, 1 - \frac{\sum_{i \neq j} s_i(\theta)}{r \cdot F(\theta)} \right] \cdot r \cdot F(q) \\ &= s_j(\theta) \cdot F(q) / F(\theta) > 0 \quad (q < p^*). \end{aligned}$$

This is unaffected by replication. \parallel

The reason behind this result is very simple. Consider the initial position where all firms are setting the competitive price, θ . If costs are strictly convex then all firms are supply constrained, so that $\pi_{rj}(\theta) = \xi_j(\theta)$. Consider a Nash deviant raising his price above θ . Under FCFS his contingent demand is given by consumers left unserved by the other firms. The number of households left over will be exactly sufficient to demand all the Nash deviant’s supply $s_j(\theta)$ were he to set θ . But this number is totally unaffected by replication, since $s_j(\theta)$ is unaffected. So no matter how large the industry, given θ_{-j} , the Nash deviant will face the same sized captive market of consumers unable to buy from the lower priced firms.

APPENDIX: PROOF OF THEOREM 3

We first need some more definitions and notation. Consider the subset B of types of firms whose supplies σ_i are bounded for prices in the interval $[\theta, p^*]$: $B = \{i \in g : \sigma_i(p) \text{ is bounded for } p \in [0, p^*]\}$. Then define the output τ_i for each firm in B :

$$\tau_i = \max [\sigma_i(p^*), \Xi] \quad (F(0) = \Xi \text{ by Assumption A2})$$

and let $Y = \max_{i \in B} \tau_i$ if B non-empty, Ξ otherwise.

In our proof we shall employ truncated supply functions to avoid algebraic difficulties when σ_i is unbounded. Define $\sigma_i^*: R_+ \rightarrow [0, Y]$ where $\sigma_i^*(p) = \text{def } \min [Y, \sigma_i(p)]$. Given the properties of σ_i , σ_i^* is bounded, non-decreasing, left-continuous, and right-lower-semi-continuous. Corresponding to this truncated supply function we define the truncated profit function $\xi_j^*(p) = p \cdot \sigma_j^*(p) - c_j(\sigma_j^*(p))$.

Four Lemmas will be employed in the Theorem. The first Lemma concerns the truncated excess-demand function $F(p) - \sum_g \sigma_i^*(p)$ which is strictly decreasing when $F(p) > 0$, and is left-continuous and right-upper-semicontinuous. All results hold for CCD and FCFs.

Lemma 1. *Let $\lambda > 0$. There exists $\delta > 0$ s.t. if $F(p) - \sum_g \sigma_i^*(p) > -\delta$ then:*

$$p < \theta + \lambda.$$

Proof. It follows immediately from left-continuity, strong monotonicity and the fact that $F(\theta) - \sum_g \sigma_i^*(\theta) \geq 0$. \parallel

Lemma 2 (Undercutting Lemma). *Let $p \in R_+^m$. If for some $p \in R_+$ and some $j \in g_r$, $\sum_{p_i \leq p} x_i \geq \sigma_j^*(p)$ ($i \in g_r$, $p_i \leq p$ may include j) then*

$$\sup \pi_{rj}(q, p_{-j}) \geq \xi_j^*(p) (q \in R_+) q.$$

Proof. For $q < p$, $d_{rj}(q, p_{-j}) > \sum_{p_i \leq p} x_i > \sigma_j^*(p) \geq \sigma_j^*(q)$ so that $\pi_{rj}(q, p_{-j}) \geq \xi_j^*(q)$. Hence

$$\sup_{q \in R_+} \pi_{rj}(q, p_{-j}) \geq \sup_{q < p} \pi_{rj}(q, p_{-j}) \geq \xi_j^*(p). \quad \parallel$$

Lemma 3. *Let $\delta > 0$, $p, \underline{x} \in R_+^m$. There exists $\varepsilon > 0$ such that if \underline{p} is an ε -equilibrium and $\underline{x} \gg 0$, and for some p and for some $h \subseteq g_r$:*

$$\sup_{q \leq p} \pi_{rj}(q, p_{-j}) \geq \xi_j^*(p) \quad \text{for all } j \in h$$

$$\text{then: } \sigma_j^*(p_j) > \sigma_j^*(p) - \delta/2n \quad \text{whenever } p_j < p, j \in h$$

Proof. Since \underline{p} is an ε -equilibrium:

$$\pi_{rj}(p_j, p_{-j}) \geq \xi_j^*(p) - \varepsilon \quad \text{for all } j \in g_r$$

A necessary condition for this to be satisfied for firms $j \in h$, $p_j < p$ is that:

$$\xi_j^*(p_j) > \xi_j^*(p) - \varepsilon \tag{a1}$$

(to see why, note that for $p_j \leq p$, $\pi_{rj} = \xi_j \geq \xi_j^*$).

σ_j^* are left-continuous and non-decreasing, so that given δ there exists $\gamma > 0$ such that for $p_j < p$,

$$\sigma_j^*(p) - \sigma_j^*(p_j) < \delta/2n \quad \text{whenever } p - p_j < \gamma \tag{a2}$$

ξ_j^* is continuous and strictly increasing. Continuity of its inverse around $\xi_j^*(p)$ ensures that given $\gamma > 0$, there exists $\varepsilon > 0$ such that

$$|p - p_j| < \gamma \quad \text{whenever } |\xi_j^*(p) - \xi_j^*(p_j)| < \varepsilon. \tag{a3}$$

Hence by choosing ε small enough so that (a3) is satisfied, (a1) implies (a2) for firms $j \in h$, $p_j \leq p$. \parallel

Lemma 4. *Let $r \in R_+$, $\underline{p} \in R_+^m$. Then for all $j \in g_r$:*

$$\sup_{q < \theta} \pi_{rj}(q, p_{-j}) = \xi_j(\theta).$$

Proof. We shall prove this for CCD. Recall the lower bound W_{rj} in Assumption A3:

$$d_{rj}(q, p_j) \geq r \cdot F(q) - \sum_{p_i \leq q} S(p_i).$$

Since $q < \theta$,

$$d_{rj}(q, p_{-j}) > r \cdot F(\theta) - \sum_{p_i < \theta} \sigma_i(\theta) > \sigma_j(\theta) > \sigma_j(q).$$

Hence

$$\sup_{q < \theta} \pi_{rj}(q, p_{-j}) = \sup_{q < \theta} \xi_j(q) = \xi_j(\theta) = \xi_j^*(\theta). \quad \parallel$$

We now prove Theorem 3. First, we show that we can choose ε and r so that all prices are below $\theta + \lambda$. Secondly, at the end of the proof we show that we can choose ε so that all prices are above $\theta - \lambda$. The first stage of the proof deals with two exhaustive cases. In case 1 the output of firms setting the highest price p_m is greater than Y , and then in case 2 the contrary.

Case 1

$$\sum_{p_j = p_m} x_i \cong Y. \tag{a4}$$

From (a4) and the definition of Y , we are able to employ Lemmas 1-3. From Lemma 2, if any firm j undercuts p_m , then it can earn at least up to $\xi_j^*(p_m)$. If \underline{p} is an ε -equilibrium, then:

$$\pi_{rj}(p_j, p_{-j}) \cong \xi_j^*(p_m) - \varepsilon \quad \text{for all } j \in g_r.$$

Given Lemma 1 choose $\delta > 0$ such that $F(\underline{p}) - \sum_g \sigma_i^*(\underline{p}) > \delta$ implies $p < \theta + \lambda$. Given (a5), from Lemma 3, there exists $\varepsilon > 0$ such that for any j such that $p_j < p_m$,

$$\sigma_j^*(p_j) > \sigma_j^*(p_m) - \delta/2n, \quad p_j < p_m. \tag{a6}$$

Turning to firms setting $p_j = p_m$, (a4) implies that x_i must satisfy:

$$p_m \cdot x_j - c_j(x_j) \cong \xi_j^*(p_m) - \varepsilon. \tag{a7}$$

Under Assumption A1 the L.H.S. of (a7) is continuous and increasing up to $\sigma_j^*(p_m)$, so by choosing ε small enough we can ensure that given $\delta > 0$,

$$x_i > \sigma_j^*(p_m) - \delta/2n, \quad p_j = p_m. \tag{a8}$$

Recalling the definition of CCD,

$$r \cdot F(p_m) \cong \sum_{p_i = p_m} x_i + \sum_{p_i < p_m} S_i(p_i) \tag{a9}$$

$$\cong \sum_{p_i = p_m} x_i + \sum_{p_i < p_m} \sigma_i^*(p_i). \tag{a10}$$

From (a6) and (a8), by choosing ε small enough, (a10) becomes:

$$r \cdot F(p_m) > \sum_{g_r} [\sigma_j^*(p_m) - \delta/2n] \tag{a11}$$

which implies

$$F(p_m) - \sum_g \sigma_j^*(p_m) > -\delta/2 \tag{a12}$$

so that from Lemma 1

$$p_m < \theta + \lambda.$$

This proves that when (a4) holds, we can choose ε and r so that all prices are less than $\theta + \lambda$. Turning to its contrary:

Case 2

$$\sum x_i < Y \tag{a13}$$

$$p_i = p_m.$$

In order to show that the Theorem holds in this case, we assume the contrary to derive a contradiction. The relevant contrary is that no matter how small $\varepsilon > 0$, and how large r , there exists an ε -equilibrium such that $\underline{x} \gg 0$ and $p_m \cong \theta + \lambda$. We are then able to derive a contradiction, which happens to be that $p_m < \theta + \lambda$.

From (a9), since $\underline{x} \gg 0$, and by assumption $p_m \cong \theta + \lambda$,

$$r \cdot F(\theta + \lambda) - \sum_{p_i < p_m} S_i(p_i) > 0 \tag{a14}$$

(a14) enables us to derive a lower bound $X > 0$ on the output of firms setting p_m , and hence an upper bound n_m on the number of these firms.

Consider the contingent demand of any firm j if it sets its price at $p_j = \theta + (\lambda/2)$. From (a14):

$$d_{rj}(\theta + \lambda/2, p_{-j}) > r \cdot [F(\theta + \lambda/2) - F(\theta + \lambda)] > 0. \tag{a15}$$

Define $r_0 = Y/[F(\theta + \lambda/2) - F(\theta + \lambda)]$.

For $r > r_0$,

$$d_{rj}(\theta + \lambda/2, p_{-j}) > Y \cong \sigma_j^*(\theta + \lambda/2). \tag{a16}$$

Thus, if \underline{p} is an ε -equilibrium, then

$$\pi_{rj}(p_j, p_{-j}) \cong \xi_j^*(\theta + \lambda/2) - \varepsilon. \tag{a17}$$

(a17) provides a lower bound on the profits of all firms for $r > r_0$, and hence a lower bound on their outputs. Turning first to firms setting $p_j = p_m$, (a17) means that for small ε we can define $X_j > 0$:

$$p^* \cdot X_j - c_j(X_j) = \xi_j^*(\theta + \lambda/2) - \varepsilon. \quad (\text{a18})$$

Define $X = \min X_j$. From (a18) for $r > r_0$ and ε small:

$$0 < X \leq x_j \leq \sum_{p_i = p_m} x_i < Y \quad \text{for all } j \in g_r. \quad (\text{a19})$$

So that the number of firms setting p_m is less than n_m :

$$n_m = Y/X. \quad (\text{a20})$$

Recalling (a10), we obtain:

$$r \cdot F(p_m) \geq \sum_{g_r} \sigma_i^*(p_i) + \sum_{p_i = p_m} x_i - \sum_{p_i = p_m} \sigma_i^*(p_i) \quad (\text{a21})$$

From (a20) $\sum_{p_i = p_m} \sigma_i^*(p_i) < n_m \cdot Y < Y^2/X$, and recalling (a19), (a21) becomes:

$$r \cdot F(p_m) > \sum_{g_r} \sigma_i^*(p_i) - \frac{(X^2 - Y^2)}{X}. \quad (\text{a22})$$

By assumption $\sigma_i^*(p_m) \geq \sigma_i^*(\theta + \lambda/3)$. If $p_i < p_m$, then $x_i = S_i(p_i)$, and $\pi_{ri}(p) = \xi_i(p_i) = \xi_i^*(p_i)$. Since ξ_i^* is continuous and strictly increasing, for ε small enough (a14) implies $p_i \geq \theta + \lambda/3$. Hence:

$$x_i = S_i(p_i) \geq S_i(\theta + \lambda/3) \geq \sigma_i^*(\theta + \lambda/3) \quad p_i < p_m \quad (\text{a23})$$

From (a22)-(a23)

$$\begin{aligned} r \cdot F(p_m) &> \sum_{g_r} \sigma_i^*(\theta + \lambda/3) - \frac{(X^2 - Y^2)}{X} \\ &> r \cdot F(\theta) - \frac{(X^2 - Y^2)}{X}. \end{aligned}$$

Hence:

$$F(p_m) - F(\theta) > -\frac{1}{r} \cdot \frac{(X^2 - Y^2)}{X}. \quad (\text{a24})$$

Since F is continuous and strictly decreasing, we can choose r large enough so that $p_m - \theta < \lambda$, the desired contradiction. Hence in both cases 1 and 2 we were able to choose ε an r so that all prices set in an ε -equilibrium with positive output were less than λ above the competitive price.

A lower bound on prices set is easily found using Lemma 4. Let $\lambda > 0$ consider any firm j . Either $\xi_j(\theta) = 0$, or $\xi_j(\theta) > 0$. If $\xi_j(\theta) = 0$, then for all prices $p_j < \theta$, $x_j = \sigma_j(p_j) = 0$, which violates the positive output requirement. If $\xi_j(\theta) > 0$, since p is an ε -equilibrium:

$$\sup_{q \in R_+} \pi_{rj}(q, p_{-j}) \geq \sup_{q < \theta} \xi_j(q) \geq \xi_j(\theta) - \varepsilon$$

Since the inverse of ξ_j is continuous, there exists ε such that

$$p_j > \theta - \lambda$$

which completes the proof. \parallel

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NOTES

1. Mixed strategies will exist in the special case of identical firms (see Dixon 1984a). However, we find mixed strategies implausible in Bertrand-Edgeworth models. Since price-setting is not irreversible, the no regret property of pure-strategy equilibria is particularly attractive.

2. The relative merits of CCD and FCFS are discussed in Dixon (1986) and Davidson and Deneker (1982). When excess demand is *anticipated* (as in this model), we observe both types of rationing. CCD avoids the costs of offending consumers by turning them away empty handed, whilst FCFS involves serving fewer people.

3. Definitions of semi-continuity. Consider the function $f: R^1 \rightarrow R$, and any sequence $\{x_n\}$ such that $x_n \rightarrow x$, x_n and x in R^1 . f is upper semi-continuous if as $n \rightarrow \infty$:

$$\text{Limsup } f(x_n) \leq f(x).$$

f is lower semi-continuous if as $n \rightarrow \infty$:

$$\text{Liminf } f(x_n) \geq f(x).$$

4. The only reason that the competitive price is an equilibrium in the standard Bertrand model is that as either firm considers raising its price above the competitive price, its demand becomes zero as the other firm meets all demand, despite the fact that its profits remain zero.

5. We could augment the firms' strategies to include the amount that they are willing to trade up to at a given price.

6. It may be objected that the Edgeworthian cost function is discontinuous at capacity (when it becomes infinite). This does not matter here because by Assumption A2 firms will never set prices above p^* . We can always construct an artificial cost function with a kink at capacity that is sufficiently large to ensure that the firm's supply does not exceed capacity for prices below p^* .

7. Manipulability is only an insuperable problem with perfect information—when the household knows firms' supply functions and industry demand. Otherwise the household knows only that it will receive some unknown proportion of its expressed demand. Thus the risk of being given what it asks for may be sufficient to place an upper bound on households expressed demands.

8. These include the usual "equal shares" assumption (Dasgupta and Maskin (1982), Dixon (1984a)); Allen and Hellwig (1983) assume that demands for firms are proportional to supplies; and Gelman and Salop's (1983) lexicographic preference assumption makes consumers prefer an incumbent's output to the entrant's at the same price.

9. "Adding up" here does *not* mean that the contingent demands for firms setting the same price must sum to the contingent demand there would be if only one firm set the price (i.e. V_{rj}). As noted in Dixon (1986), contingent demands need not satisfy Walras' Law. To see this, consider a duopoly where both firms set the same price and supply nothing: each firm's contingent demand will equal the industry demand, since that is how much it *could* sell given that the other firm has zero supply. What do need to "add up" are the *actual* trades, which depend on demands and supplies: this is what condition A3(b) captures (the L.H.S. are actual trades, the R.H.S. the total trades determined by the min condition).

10. There are two meanings to the term " ϵ " or "approximate" equilibrium. The alternative to definition 3 is that each players' *action* is within ϵ of his best response (the approximation is within the strategy space, rather than payoffs).

11. For a rather ingenious example of existence due to demand being perfectly elastic at θ see Shubik (1955, p. 425-425).

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