

INTRODUCTORY MATHEMATICS FOR ECONOMICS MSC'S.
LECTURE 1: DIFFERENTIATION.

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1. Some Definitions.

- An Integer: A positive or negative whole number, as in “counting. -3,-2,-1,0,1,2,3
- A rational number: the ratio of two integers. $\frac{11}{23}, \frac{-2}{3}$
- An irrational number: cannot be represented as the ratio of two integers. E.g. $\sqrt{2}, \pi \dots$
- **Real Numbers and the Real line \mathfrak{R} :** the set of all rational and irrational numbers. This is a “continuum”:
 - $\mathfrak{R} = \{x: -\infty < x < \infty\} = (-\infty, \infty)$
 - $\mathfrak{R}_+ = \{x: 0 \leq x < \infty\} = [0, \infty)$
 - Open interval: includes end-points. $(0,1)$ means strictly greater than 0 and strictly less than 1
 - Closed interval: includes end-points. $[0,1]$.
 - Half open interval: includes one of endpoints, but not the other. $[0,1)$ includes 0, but not 1.
- In economics, use real numbers most of the time: x might be output, price, advertising expenditure etc. This is an approximation (cannot set a price π), but as in physics, seems to work well.

2. Functions f.

- takes a real number (input) and transforms it into another real number (output).
- $y = f(x)$. You take a real number input x and it is *mapped onto* output y .

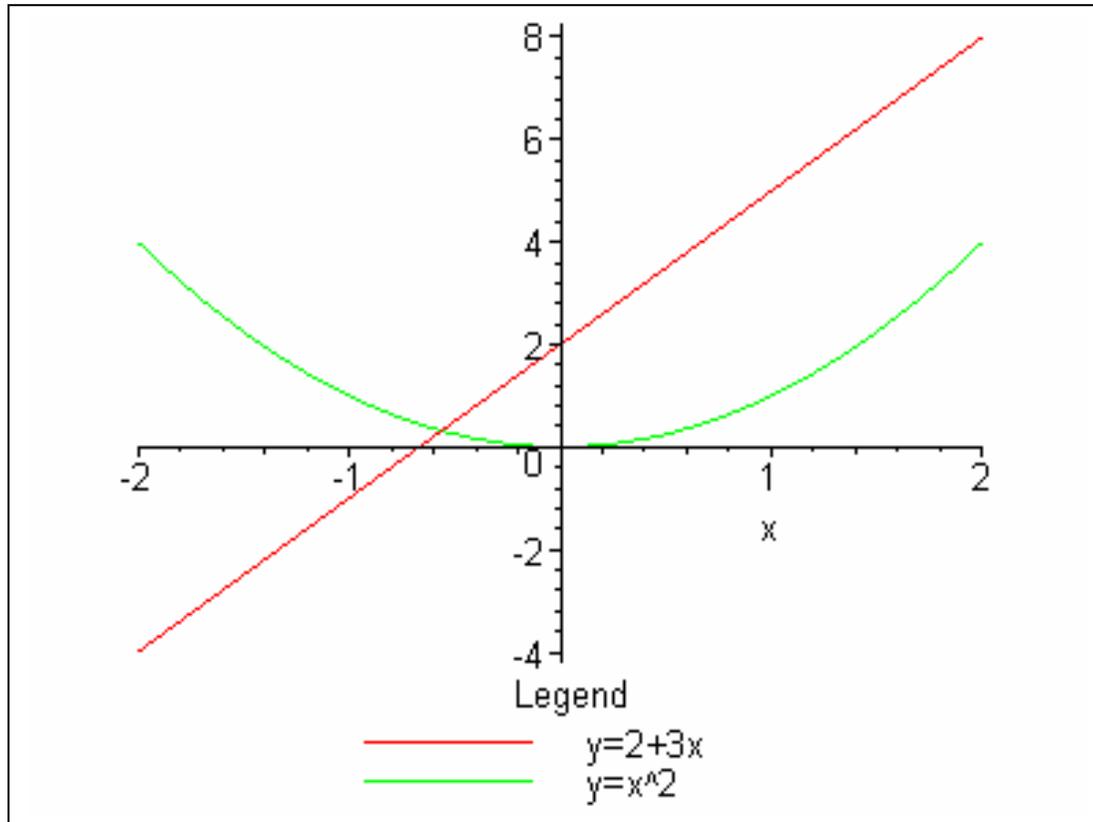
- $f(x): y=2+(3.x)$

x	$y=f(x)$
-10	-28
10	32
1.2	5.6

- $f(x): y=x^2$

x	$y=f(x)$
-10	100
10	100
1.2	1.44

- A function is **1-1** if each input has a unique output, and each output has a unique input. If we look above, $y=x^2$ is not 1-1. the output 100 can be generated by two values of x : +10 and -10.



$y=2+3x$ is **monotonic**: it is always strictly increasing and y gets larger as x gets larger.

It is also 1-1. Each value of y is associated with one value of x and vice versa.

$y=x^2$ is **non-monotonic**. For $x < 0$ y is decreasing in x ; for $x > 0$ it is increasing in x .

it is not 1-1: for each value of $y \neq 0$ there are two values of x which generate it, the positive and negative square roots.

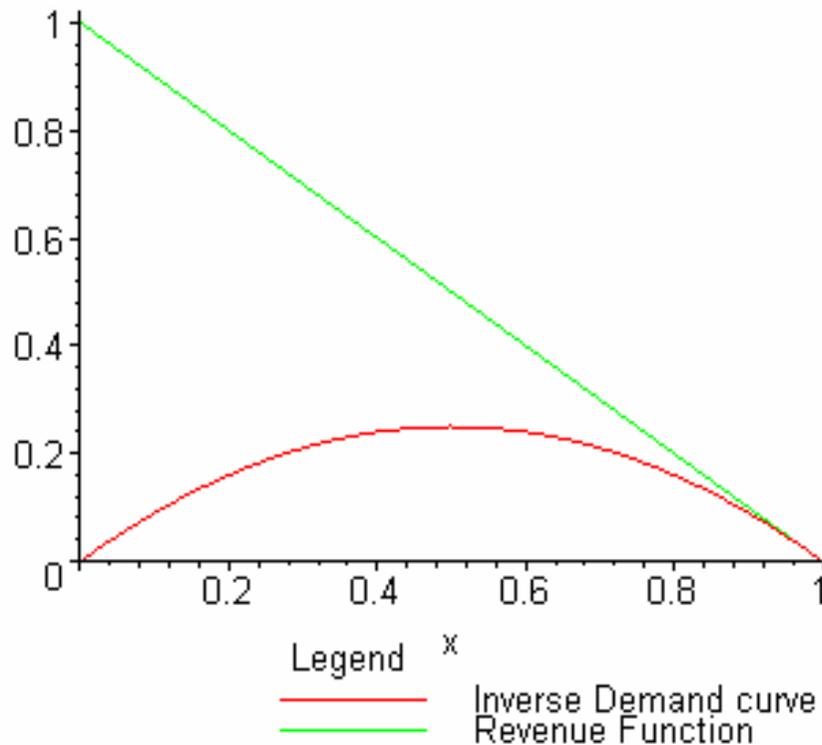
$y=2+3x$ is invertible: we can write y as a function of x and x as a function of y :

$$x = \frac{y}{3} - \frac{2}{3}$$

Demand curve: $X = 1 - P$ X =output/quantity: P =price

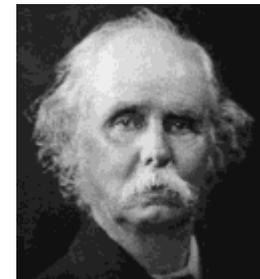
Inverse Demand curve: $P = 1 - X$

Revenue Function: $R = X.P = X.(1 - X) = X - X^2$



In Economics, we usually deal with the *inverse* demand curve, and treat price as a function of Quantity (Alfred Marshall, British economist in late C19th)

That is why price is on the vertical axis!



3. Differentiation made simple.

Differentiation was invented by Sir Isaac Newton, the great British 17th Century Physicist (*Principia Mathematica*).

If you take a function and differentiate it, you get the SLOPE of the function.

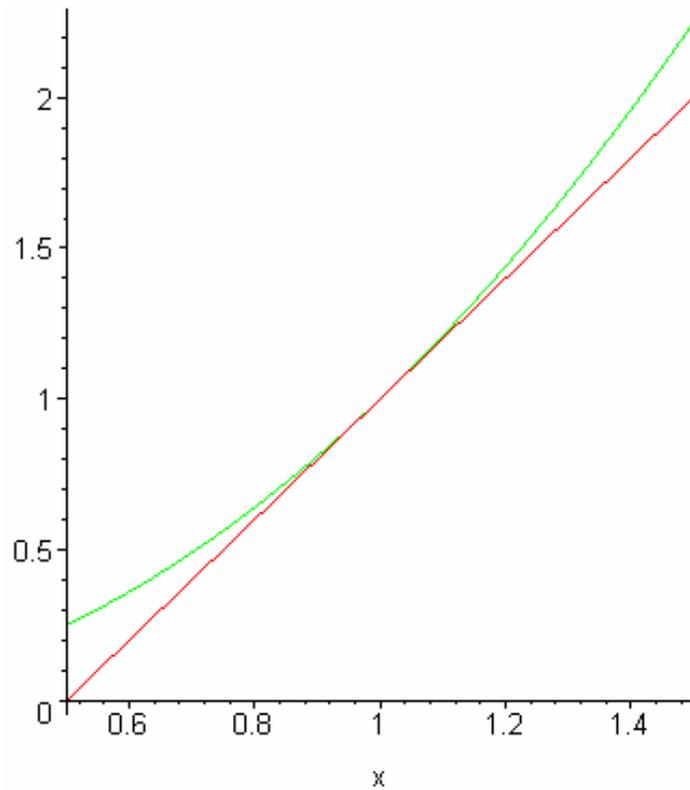
–the change in y is called Δy , the change in x is Δx . The ratio of the changes is $\frac{\Delta y}{\Delta x}$

The derivative is the ratio $\frac{\Delta y}{\Delta x}$ as Δx becomes very small.

For example: $y = x^2$ at $x=1, y=1$. As Δx becomes very small, the ratio is $\frac{dy}{dx} = 2$.

Δx	Δy	$\frac{\Delta y}{\Delta x}$
1 (x changes from 1 to 2)	3 (1 to 4)	3
0.1 (x changes from 1 to 1.1)	0.21 (1 to 1.21)	2.1
0.01 (1 to 1.01)	0.0201 (1 to 1.0201)	2.01
0.001	0.002001 (1 to 1.002001)	2.001

This is also called the *tangent* or *slope* of the function



At the point $y=1$, $x=1$ the green line is $y = x^2$.

The red line is the tangent at that point $(1,1)$ which has the slope $\frac{dy}{dx} = 2$.

The Rules of Differentiation:

Rule 1: Power rule.

$$y = ax^n$$

$$\frac{dy}{dx} = anx^{n-1}$$

$$y = x^2; f' = 2x$$

Examples: $y = 3x^5; f' = 15x^4$

$$y = 2x^{-8}; f' = -16x^{-9}$$

Note: notation $\frac{dy}{dx} \equiv f'$

Rule 2: Natural Logarithm.

$$y = \ln ax$$

$$\frac{dy}{dx} = \frac{1}{x}$$

$$y = \ln 3x$$

Example $\frac{dy}{dx} = \frac{1}{x}$

Rule 3: Exponential Function.

$$y = e^{ax}$$

$$\frac{dy}{dx} = ae^{ax}$$

Example $y = e^x; f' = e^x$
 $y = e^{-x}; f' = -e^{-x}$

These are the basic rules used in economics: we do not need to use trigonometric functions (sin, cosine etc) except for very specialised uses.

We then combine Rules 1-3 with further rules: these deal with differentiation of combinations of *two functions* $f(x)$ and $g(x)$. The aim is to take a complex expression (equation) and break it down into simpler parts f and g . Rules 1-3 can be used to differentiate f and g themselves,

Rule 4: Product rule.

$$y = f(x).g(x)$$

$$\frac{dy}{dx} = f'.g + g'.f$$

Examples:

(a)

$$y = x^2 e^{-x}$$

This is the function.

$$f(x) = x^2; f' = 2x$$

We break it up into two simple functions

$$g(x) = e^{-x}; g' = -e^{-x}$$

We apply rules 1 and 3 to the simple functions

$$\frac{dy}{dx} = f'g + g'f$$

We then apply the product rule 4 to the two simple functions

$$= 2xe^{-x} - e^{-x}x^2 = xe^{-x}[2 - x]$$

(b)

$$y = 4x^4$$

$$f(x) = 4x; f' = 4$$

$$g(x) = x^3; g' = 3x^2$$

$$\frac{dy}{dx} = f'g + g'f = 4x^3 + 4x \cdot 3x^2 = 4x^3 + 12x^3 = 16x^3$$

we know the answer here: but the product rule gives us the same answer as the rule 1: it shows the rules are consistent!

(c)

$$y = x \ln x$$

$$f(x) = x; f' = 1$$

$$g(x) = \ln x; g' = x^{-1}$$

$$\frac{dy}{dx} = f'g + g'f = \ln x + x \cdot x^{-1} = \ln x + 1$$

Rule 5: Function of a function (Chain rule)

This is a widely used rule: you break down the expression so that there are two or more stages.

$$y = f(z); z = g(x)$$

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = f' \cdot g'$$

Examples:

(a)

We start with:

$$y = \ln x^2$$

Break it down into two expressions
we know how to differentiate

$$y = \ln z$$

$$z = x^2$$

Apply the rule

$$\frac{dy}{dx} = \frac{dy}{dz} \frac{dz}{dx} = \frac{1}{z} 2x = \frac{2}{x}$$

(b)

$$y = e^{x^2}$$

$$y = e^z$$

$$z = x^2$$

$$\frac{dy}{dx} = \frac{dy}{dz} \frac{dz}{dx} = e^z 2x = 2xe^{x^2}$$

(c) reciprocals

$$y = \frac{1}{x^2}$$

$$y = z^{-1}; z = x^2$$

$$\frac{dy}{dx} = \frac{dy}{dz} \frac{dz}{dx} = -z^{-2} \cdot 2x = -\frac{2x}{x^4}$$

$$= -\frac{2}{x^3}$$

(e) exponential $y = e^{(ax-b)^2}$

$$y = e^z$$

$$z = w^2$$

$$w = (ax - b)$$

$$\frac{dy}{dx} = \frac{dy}{dz} \frac{dz}{dw} \frac{dw}{dx} = e^z 2wa = e^{(x-a)^2} 2a(ax - b)$$

The last rule is actually implied by the previous two rules, but we state it for convenience!

Rule 6: quotient rule.

$$y = \frac{u(x)}{v(x)}$$

$$\frac{dy}{dx} = \frac{v \cdot u' - u \cdot v'}{v^2}$$

For example:

$$y = \frac{e^{2x}}{3x^2}$$

$$u = e^{2x}; u' = 2e^{2x}$$

$$v = 3x^2; v' = 6x$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{v \cdot u' - u \cdot v'}{v^2} = \frac{6x^2 e^{2x} - 6x e^{2x}}{9x^4} \\ &= e^{2x} \frac{2(x-1)}{3x^3} \end{aligned}$$

To summarise:

You take function. You use rules 4-6 to break down the expression into simple parts that you can differentiate using rules 1-3.

4 Economics applications.

(a) *Revenue and marginal revenue.*

The inverse demand curve:

$$P = P(x)$$

(Price depends on output).

Revenue: price times output:

$$R(x) = x.P(x)$$

MR: differentiate using produce rule

$$\frac{dR}{dx} = xP' + P$$

$$= P\left(1 + \frac{xP'}{P}\right)$$

Rearrange, using elasticity formula

$$\varepsilon = -\frac{P}{x} \frac{1}{P'} = -\frac{P}{x} \frac{dx}{dP}$$

$$= P\left(1 - \frac{1}{\varepsilon}\right)$$

Let us look at some particular demand curves: these are ones you will see used in economics often!

Linear Demand.

The Demand curve:

$$x = a - P$$

a is the “intercept” term. It is the price when output is zero.

$$P(x) = a - x$$

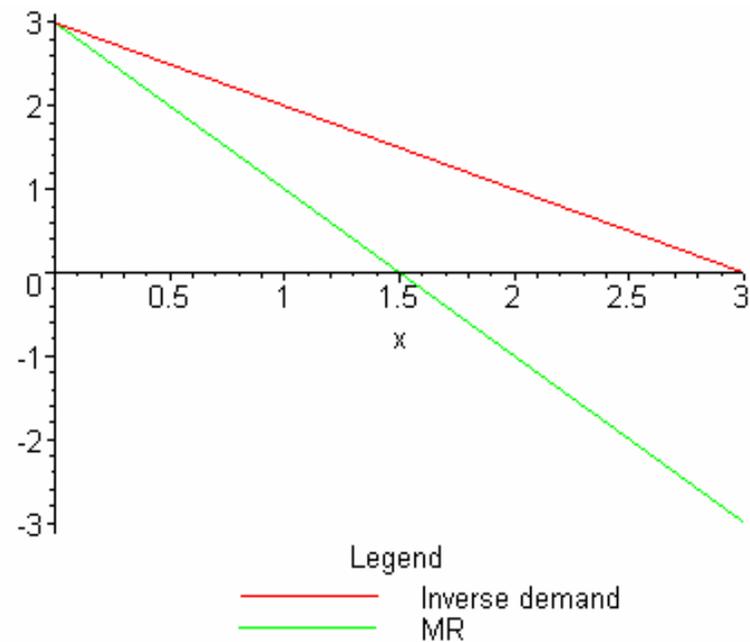
$$R(x) = x.(a - x)$$

$$\begin{aligned} \frac{dR}{dx} &= xP' + P = -x + (a - x) \\ &= a - 2x. \end{aligned}$$

Note that marginal revenue always has a slope that is twice that of the linear demand curve.

Take the example of $a=3$.

What happens to the elasticity along the linear demand curve?



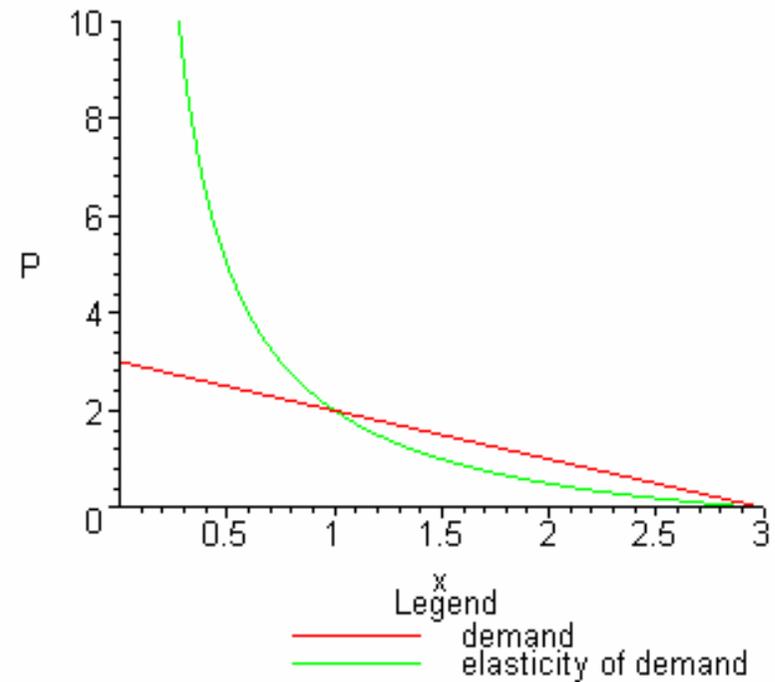
$$P = a - x;$$

$$\begin{aligned}\varepsilon &= -\frac{P}{x} \frac{dx}{dP} \\ &= \frac{(a-x)}{x}\end{aligned}$$

As output increases, the elasticity increases;

$$\frac{d\varepsilon}{dx} = \frac{-x - (a-x)}{x^2} = -\frac{a}{x^2} < 0$$

If $x=0$ ($P=a$), $\varepsilon=\infty$. If $x=a$ ($P=0$) $\varepsilon=0$.



Demand curve 2: constant elasticity of demand.

Direct demand

$$x = aP^{-\varepsilon}$$

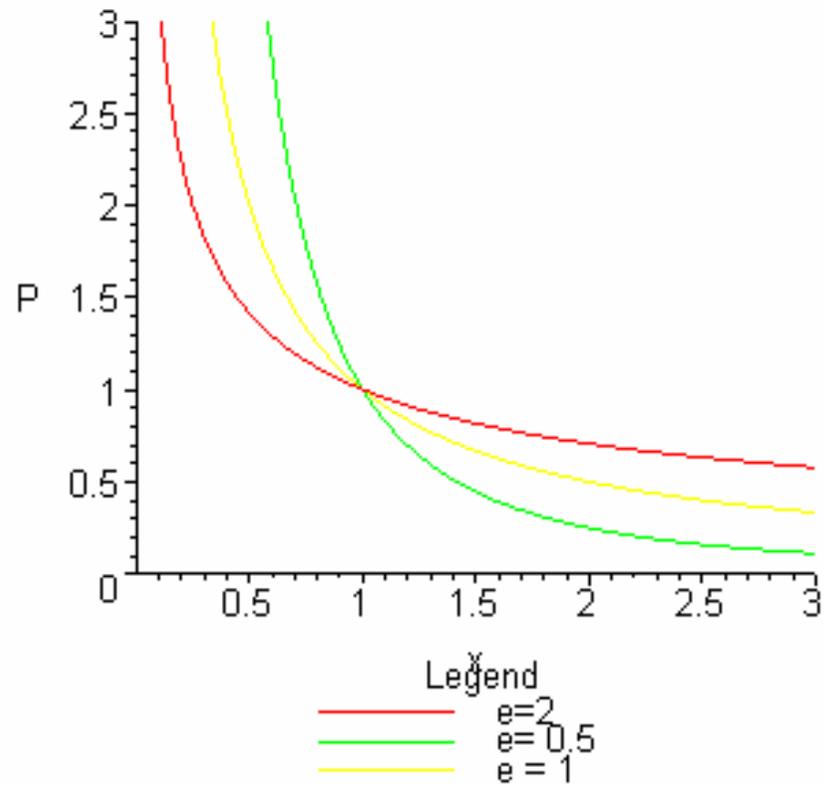
Inverse Demand

$$P = \left(\frac{x}{a} \right)^{-\frac{1}{\varepsilon}}$$

Elasticity.

$$\varepsilon = -\frac{P}{x} \frac{dx}{dP} = \frac{P}{aP^{-\varepsilon}} \cdot a\varepsilon P^{\varepsilon-1} = \varepsilon$$

If $\varepsilon=1$, then we have a rectangular hyperbola: $P \cdot x = a$.



The Yellow line represents the unit elastic demand curve. Total expenditure $P.X$ is constant.

The red line represents an elastic demand curve: revenue is decreasing with price.

The green line represents an inelastic demand curve: revenue is increasing with price.

$$R(x) = x \cdot \left(\frac{x}{a} \right)^{-\frac{1}{\varepsilon}}$$

Revenue: output times price

$$= a^{\frac{1}{\varepsilon}} x^{1-\frac{1}{\varepsilon}}$$

Revenue as a function of output.

$$MR = a^{\frac{1}{\varepsilon}} \left(\frac{\varepsilon - 1}{\varepsilon} \right) x^{-\frac{1}{\varepsilon}}$$

Marginal Revenue: zero if $\varepsilon=1$ (yellow line), strictly positive if $\varepsilon>1$ (red line) and strictly negative if $\varepsilon<1$ (green line).

Cost functions.

$C(x)$ Total cost of producing output x . For example:

$$C(x) = F + \frac{c}{2} \cdot x^2$$

F : overhead cost, does not depend on output.

c is marginal cost (MC): $\frac{dC}{dx} = c$

Average Cost:

$$AC(x) = \frac{C(x)}{x}$$

Average cost is total output divided by output.

$$\frac{dAC}{dx} = \frac{x C' - C}{x^2} = \frac{1}{x} \left(C' - \frac{C}{x} \right)$$

The Change in AC as output increases (slope of AC).

$$= \frac{1}{x} (MC - AC)$$

AC is decreasing if $MC < AC$; increasing if $MC > AC$,

For example, take the previous cost function with $F=2$, $c=1$

$$C(x) = 2 + \frac{1}{2} \cdot x^2$$

$$\frac{dAC}{dx} = 2 \left[\frac{1}{4} - \frac{1}{x^2} \right]$$

$$MC = x$$

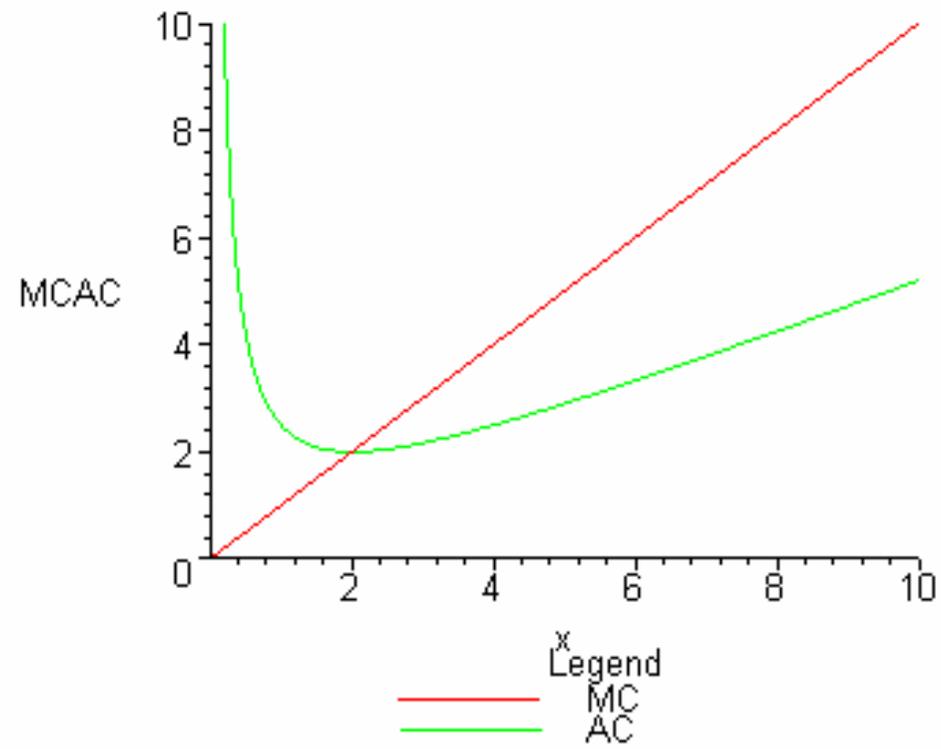
$$AC = \frac{2}{x} + \frac{1}{2}x$$

$$\frac{dAC}{dx} = -\frac{2}{x^2} + \frac{1}{2}$$

$$x > 2 \Rightarrow \frac{dAC}{dx} > 0 \quad \text{And } MC > AC$$

$$x < 2 \Rightarrow \frac{dAC}{dx} < 0 \quad MC < AC$$

$$x = 2 \Rightarrow \frac{dAC}{dx} = 0 \quad MC = AC$$



Thus “efficient” least cost production occurs when $AC=MC$.

Concavity and Convexity.

The second derivative of a differentiable function is obtained by differentiating the first derivative:

$$\frac{d^2 y}{dx^2} = \frac{d}{dx} \left(\frac{dy}{dx} \right)$$

$y=f(x)$: the first derivative is written as f' , the second derivative as f''

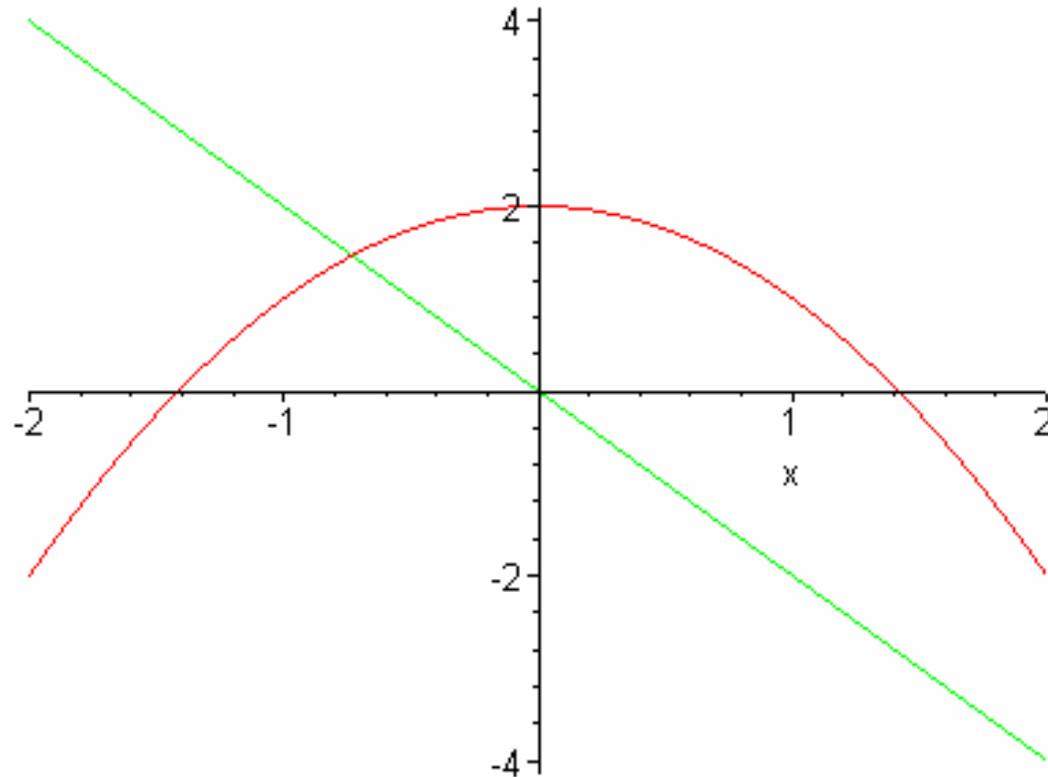
$$y = x^2 \quad \frac{dy}{dx} = 2x \quad \frac{d^2 y}{dx^2} = 2$$

A function is strictly concave if for all x $\frac{d^2 y}{dx^2} = f'' < 0$.

This means that the slope of the function is decreasing (becoming less positive, more negative).

Example: The quadratic function

$$\begin{aligned} y &= A - x^2 \\ f' &= -2x \\ f'' &= -2 < 0 \end{aligned}$$



The red line is the quadratic $y = 2 - x^2$; the green line is the first derivative $f' = -2x$. This is *decreasing* in x $f'' = -2 < 0$.

A concave function has the following shape: if it is increasing in x , the slope is becoming less steep as x increases (see $x < 0$ above); if it is decreasing in x , the slope becomes more negative (steeper – see $x > 0$ above). A concave function is like an *upside down bow* or *Hill*...

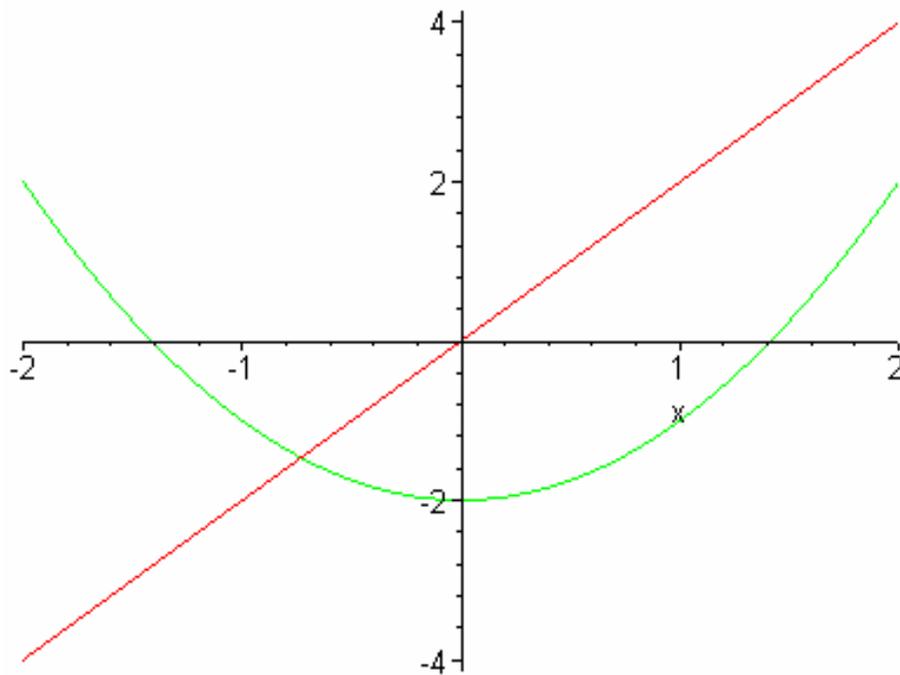
A function is strictly convex if the slope (derivative) is increasing in x .

Take the quadratic

$$y = A + x^2$$

$$f' = 2x$$

$$f'' = 2 > 0$$



The green line is the quadratic ($A=-2$): the red the first derivative.

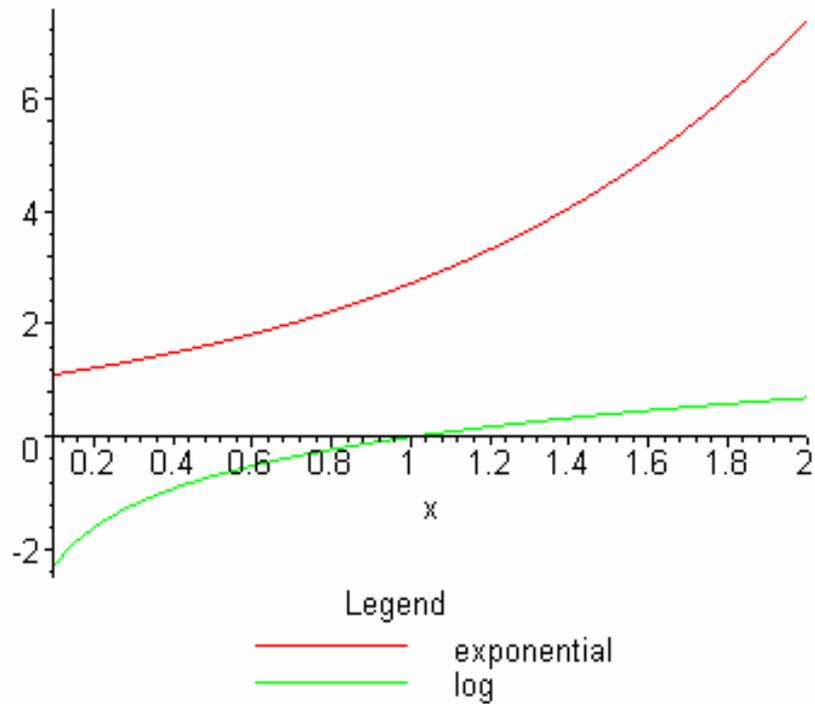
A convex function is like a bowl.

Note that the negative of a concave function is convex: if $f(x)$ is strictly convex, then $-f(x)$ is strictly concave and vice versa.

Other functions:

$$e^{ax} \Rightarrow \frac{dy}{dx} = ae^{ax} \Rightarrow \frac{d^2y}{dx^2} = a^2e^{ax} > 0 \quad \text{Exponential is convex}$$

$$\ln ax \Rightarrow \frac{dy}{dx} = \frac{1}{x} > 0 \Rightarrow \frac{d^2y}{dx^2} = -\frac{1}{x^2} < 0 \quad \text{Log is concave.}$$



Linear Functions are neither strictly convex nor strictly concave: the second derivative is always zero (the slope is always the same)

$y = a + bx$ a is the intercept term

$\frac{dy}{dx} = b$ This slopes up if $b > 0$, down if $b < 0$.

$$\frac{d^2 y}{dx^2} = 0$$

We say that a function is weakly concave if its second derivative is always non-positive: this allows it to be zero. A strictly concave function has a strictly negative second derivative, Likewise strictly convex has $f'' > 0$ and a weakly convex function $f'' \geq 0$.

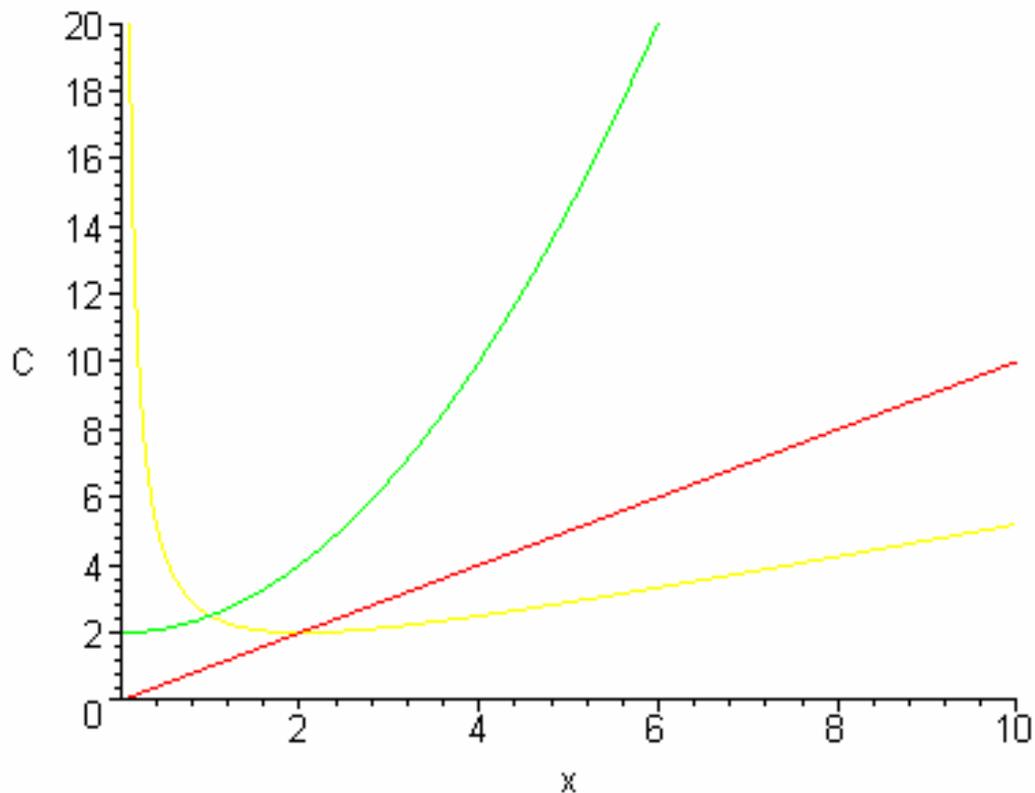
A linear function is both weakly concave and weakly convex!

Economics Examples:

Cost Function:

$$C(x) = 2 + \frac{1}{2}x^2; \quad C' = x; \quad C'' = 1 > 0.$$

In economics, we usually assume that cost functions are strictly convex.



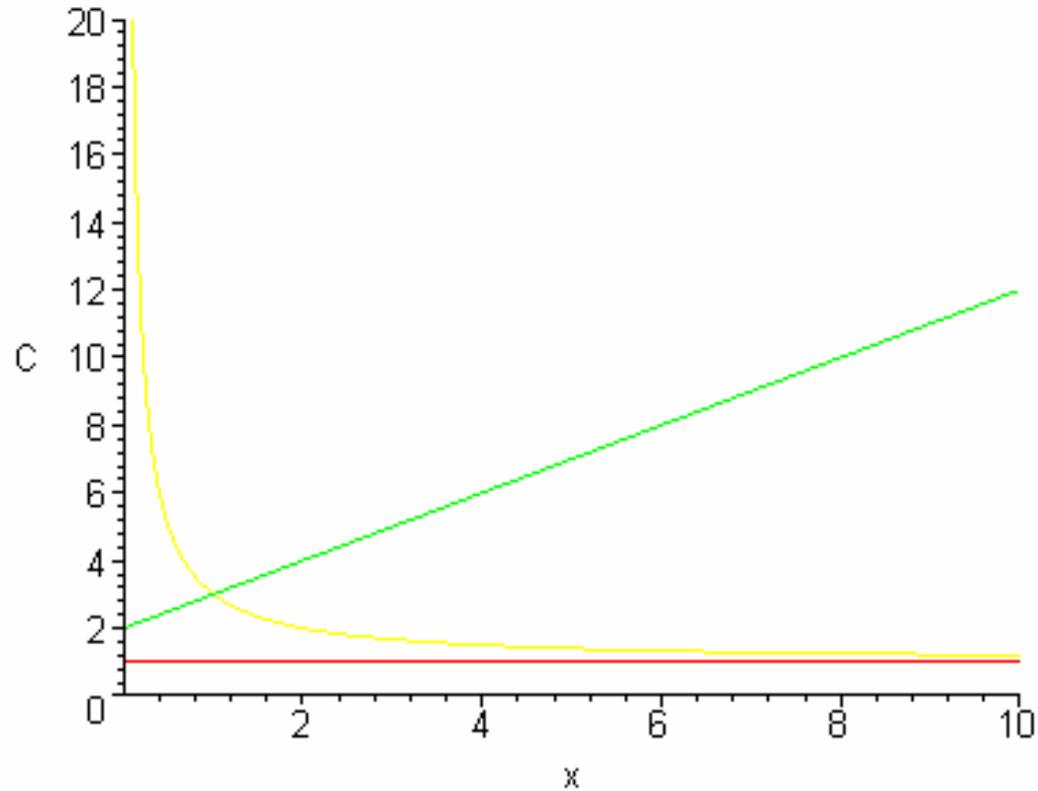
Total cost is the green line

Marginal cost the red: this is always increasing, so the function is strictly convex.

Yellow AC.

Sometimes in economics we have a linear cost function:

$$C(x) = 2 + x; \quad C' = 1; \quad C'' = 0.$$



Green: total cost.

Red: constant MC

Yellow AC, always falling ($MC < AC$)

This has increasing returns to scale (AC always falling).

Demand and revenue functions.

Linear Demand:

$$P(x) = a - x$$

Revenue

$$R(x) = x.(a - x) = ax - x^2$$

MR is decreasing in x:

$$R' = a - 2x$$

Revenue is strictly concave in:
output x.

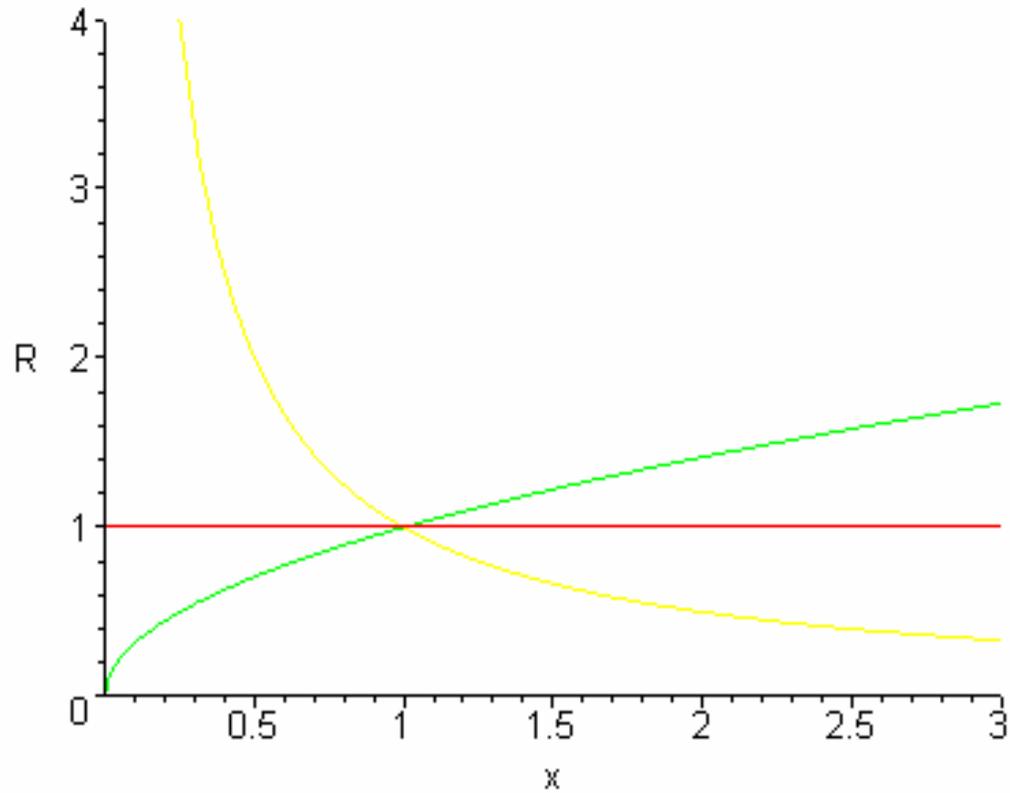
$$R'' = -2 < 0$$

Constant Elasticity:

$$R(x) = x \cdot \left(\frac{x}{a}\right)^{-\frac{1}{\varepsilon}} = a^{\frac{1}{\varepsilon}} x^{1-\frac{1}{\varepsilon}}$$

$$R' = a^{\frac{1}{\varepsilon}} \left(\frac{\varepsilon - 1}{\varepsilon}\right) x^{-\frac{1}{\varepsilon}} \quad \text{Marginal revenue positive (negative) as } \varepsilon > 1 \text{ (} \varepsilon < 1 \text{)}$$

$$R'' = -a^{\frac{1}{\varepsilon}} \left(\frac{\varepsilon - 1}{\varepsilon^2}\right) x^{-\frac{1}{\varepsilon}-1} \quad \text{Revenue is strictly concave (convex) as } \varepsilon > 1 \text{ (} \varepsilon < 1 \text{)}$$



$$P = (x)^{-\frac{1}{\varepsilon}}$$

Red line: $\varepsilon=1$

$$R(x) = x^{1-\frac{1}{\varepsilon}}$$

Green line: strictly concave and increasing: $\varepsilon>1$

Yellow line: strictly convex and decreasing: $\varepsilon<1$

Summary: What we have learned.

1. How to differentiate. Break down a complex expression into simple parts and differentiate using the rules.
2. Second derivatives and how they relate to concavity and convexity.
3. Applications to economics: demand and revenue functions; cost functions.
4. Some properties of functions: any strictly monotonic function is invertible and 1-1.

Now, get practising and apply the lessons to some economic problems!

Problem Set 1: Differentiation.

1. Find the first and second derivatives of the following functions.

$$(a) \quad x = 3p^{-2} \quad (b) \quad y = \frac{1}{\sqrt{2\pi}} e^{-(x-2)^2} \quad (c) \quad y = ax^{\frac{1-\varepsilon}{\varepsilon}}$$

$$(d) \quad y = \frac{x}{1+x} \quad (e) \quad y = \frac{1}{1+e^{-x}} \quad (d) \quad y = x^2 e^{-x}$$

2. Given the answers to part 1, can you state for each function:

(a) when and if it is strictly monotonic (where this depends on a parameter, explain);

(b) when is it strictly convex or concave or neither.

3. A firm has the demand curve $P = 5 - \frac{1}{2}x$. and the (total) cost function $C = x$.

- (a) what is the profit of the firm as a function of its output?
 - (b) show the effect of an increase in output on the profits of the firm.
 - (c) Show that profits a strictly concave function of output.
 - (d) Find the level of output where the derivative of profits with respect to output is zero.
 - (e) Show that at the point (d), marginal revenue equals marginal cost.
4. Consider a firm with a constant elasticity demand curve $x = P^{-\varepsilon}$ and cost function $C = 1 + 3x$.
- (a) what is the profit of the firm as a function of its output?
 - (b) show the effect of an increase in output on the profits of the firm.
 - (c) Show that profits a strictly concave function of output when $\varepsilon > 1$.
 - (d) What output would the firm choose to produce if $\varepsilon \leq 1$?